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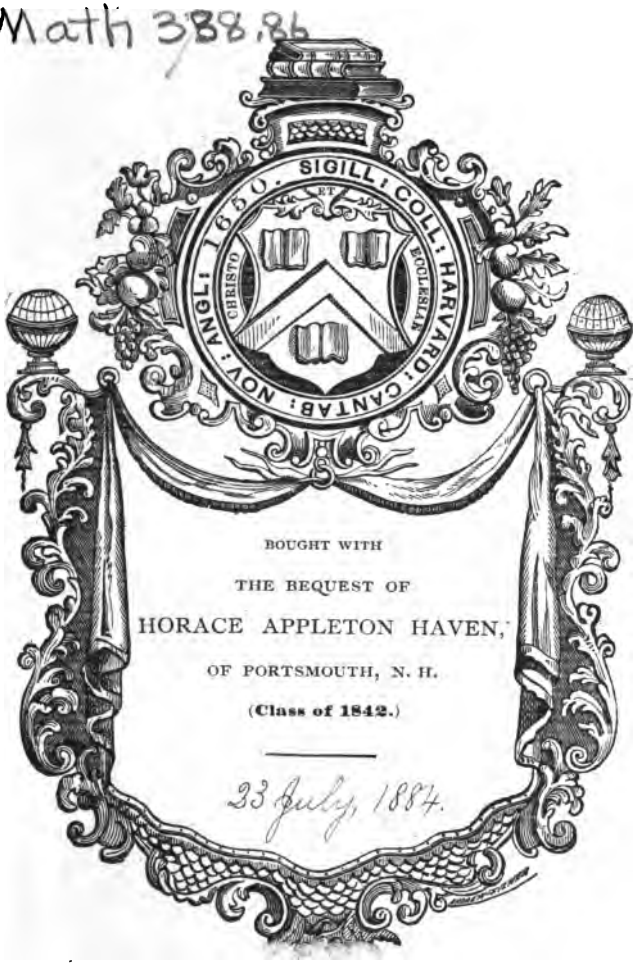

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MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

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2032. (N'Importe.)—To investigate the expression for the moment of inertia of a triangle about one of its angular points. 47

2600. (The Editor.)—Show that (1) the area of a parabolic section of a right circular cone whose slant height is unity and semi-vertical angle α, made by a plane at a distance z from the vertex along the slant side, is $\frac{8}{15}z^{\frac{5}{2}}(1-z)^{\frac{3}{2}}\sin\alpha$; (2) the greatest parabola of the series has an area of $\cos\frac{1}{3}\pi\sin\alpha$ (or $\cdot866\sin\alpha$), and its vertex at a distance of one-fourth of the slant height from the vertex of the cone; (3) if the cutting planes be uniformly distributed, the mean area of all these parabolas is $\frac{1}{6}\pi\sin\alpha$ (or $\cdot524\sin\alpha$); (4) if x be the perpendicular distance of the cutting plane from the vertex of the cone, and θ the inclination of this perpendicular to the axis of the cone, the area of a complete elliptic section is $\pi x^2\sin^2\alpha\cos\alpha\sec^{\frac{3}{2}}(\theta+\alpha)\sec^{\frac{3}{2}}(\theta-\alpha)$; (5) if $\alpha<\frac{1}{12}\pi$, and the section reaches to the base of the cone, the ellipse is a maximum when θ is the greater of the two acute-angled roots of the equation $\sin 2\theta=2\sin 2\alpha$, and a minimum when θ is the less of these roots; but if $\alpha=$ or $>\frac{1}{12}\pi$ the ellipse increases continuously from coincidence with the edge to coincidence with the base of the cone; (6) the eccentricity of any section is $\sec\alpha\sin\theta$, which is a maximum when the cutting plane is parallel to the axis; (7) if the cutting planes be equably distributed in space, the mean eccentricities of the ellipses, the hyperbolas, and all the sections of the cone, are respectively

$$\frac{2\cos^3\alpha+2\tan\alpha(2-3\sin\alpha+\sin^3\alpha)}{3\cos\alpha\cos 2\alpha+3(\frac{1}{2}\pi-\alpha)\sin\alpha},\quad \frac{2\tan\alpha(5+\cos 2\alpha)}{3(2\alpha+\sin 2\alpha)},$$

$$\frac{2\cos^3\alpha+2\tan\alpha(2+3\sin\alpha-\sin^3\alpha)}{3\cos\alpha+3(\frac{1}{2}\pi+\alpha)\sin\alpha},$$

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which, when the cone is right-angled at the vertex, become

$$\frac{16}{3\pi}(\sqrt{2}-1) = \cdot 703, \quad \frac{20}{3(2+\pi)} = 1\cdot 297, \quad \frac{24+16\sqrt{2}}{12+9\pi} = 1\cdot 158;$$

(8) the respective probabilities that a random plane will cut a complete ellipse, an ellipse of any kind, or an hyperbola, are

$$\frac{\cos\alpha - (\frac{1}{2}\pi - \alpha)\sin\alpha}{\cos\alpha + (\frac{1}{2}\pi+\alpha)\sin\alpha}, \quad \frac{\cos\alpha\cos 2\alpha + (\frac{1}{2}\pi-\alpha)\sin\alpha}{\cos\alpha+(\frac{1}{2}\pi+\alpha)\sin\alpha}, \quad \frac{2\cos\alpha\sin^2\alpha + 2\alpha\sin\alpha}{\cos\alpha+(\frac{1}{2}\pi+\alpha)\sin\alpha},$$

which, for a right-angled cone, become

$$\frac{4-\pi}{4+3\pi} = \cdot 064, \quad \frac{\pi}{4+3\pi} = \cdot 234, \quad \frac{4+2\pi}{4+3\pi} = \cdot 766;$$

(9) the mean area of all the complete ellipses is

$$\frac{\frac{3}{2}\pi\sin^2\alpha\cos\alpha}{\cos\alpha-(\frac{1}{2}\pi-\alpha)\sin\alpha}\int_0^{\frac{1}{2}\pi-\alpha}\frac{\cos^{\frac{1}{2}}(\theta+\alpha)}{\cos^{\frac{1}{2}}(\theta-\alpha)}\sin\theta\,d\theta;$$

(10) in a right-angled cone, the mean area of the complete ellipses is

$$\frac{(2\pi)^{\frac{1}{2}}}{12(4-\pi)}\left(\left\{\Gamma(\tfrac{1}{4})\right\}^2 + 12\left\{\Gamma(\tfrac{3}{4})\right\}^2\right) - \frac{2\pi}{4-\pi} = \cdot 26409. \ldots \quad 17$$

3325. (A. B. Evans, M.A.)—Find general values of x, y, z such that, when a is greater than unity,

$$x^2+axy+y^2 = \square, \quad x^2+axz+z^2 = \square, \quad y^2+ayz+z^2 = \square. \ldots \quad 45$$

3577. (S. Watson.)—From the extremity A of the major axis of a given ellipse a chord AP is drawn, and another chord PQ perpendicular to AP. Find the equation and area of the locus of the middle point of PQ..... 105

3691. (Dr. Hart.)—In a plane triangle, it is required to find integers for the sides, diameter of the inscribed circle, and the side of the inscribed square, such that the sum of the diameter and side shall be a square. 103

3752. (Asher B. Evans, M.A.)—Divide unity into three positive parts such that the sum of their squares, and also the sum of their cubes, shall be square numbers. 100

3790. (S. Watson.)—From a fixed point P on the circumference of a circle, a ball whose elasticity is e is struck so as to impinge on the circle at two other points Q and R, and then return to P. Find the average area of the triangle PQR for all values of e. 57

3803. (Rev. Dr. Booth, F.R.S.)—If $f(x, y) \equiv \mathrm{F} = 0$ be the projective equation of a plane curve, show that the tangential equation of its evolute may be found by eliminating x and y from the equations $\xi = \frac{d\mathrm{F}}{dy} \div \left(\frac{d\mathrm{F}}{dy}x - \frac{d\mathrm{F}}{dx}y\right)$, $v = -\frac{d\mathrm{F}}{dx} \div \left(\frac{d\mathrm{F}}{dy}x - \frac{d\mathrm{F}}{dx}y\right)$, $f(x, y) = 0$; and (2) show therefrom that the tangential equation to the evolute of the ellipse is $a^2v^2 + b^2\xi^2 = (a^2-b^2)\,\xi^2v^2$. ... 53

3895. (J. Griffiths, M.A.)—If F denote a homogeneous function of the degree n, show that the locus of the foci of an inscribed conic $(\lambda\alpha)^{\frac{1}{2}} + (\mu\beta)^{\frac{1}{2}} + (\nu\gamma)^{\frac{1}{2}} = 0$, where λ, μ, ν satisfy the relation $\mathrm{F}(\lambda, \mu, \nu) = 0$, is an nth circular curve of the degree $3n$ given

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by the equation F (X, Y, Z) = 0; X, Y, Z being, for shortness, respectively written for

$x(y^2+z^2+2yz\cos A)$, $y(z^2+x^2+2zx\cos B)$, $z(x^2+y^2+2xy\cos C)$. 53

3958. (Professor Wolstenholme.)—1. Prove that the squares of the semi-axes of the conic of closest contact at any point of a curve are the values of x given by the equation

$$x^2(9+b^2-3ac)^3-9a^2(18+2b^2-3ac)(9+b^2-3ac)x+729a^4=0,$$

and the product of the focal distances is $9a^2\div(9+b^2-3ac)$; a, b, c being the values of ρ, $\frac{d\rho}{ds}$, $\frac{d^2\rho}{ds^2}$ at the point, where ρ is the radius of curvature and s the arc.

2. Hence prove that the eccentricity e is given by the equation

$$9\frac{(e^2-2)^2}{1-e^2}=\frac{(18+2b^2-3ac)^2}{9+b^2-3ac};$$

and deduce the conditions that the conic may be a circle.

3. Prove that the equation of the conic of closest contact at any point of a given curve referred to the tangent and normal at the point as co-ordinate axes is $Ax^2+2Bxy+Cy^2=2y$; the values of A, B, C being

$$\frac{1}{\rho},\quad -\frac{1}{3\rho}\frac{d\rho}{ds},\text{ and }\frac{1}{\rho}+\frac{2}{9\rho}\left(\frac{d\rho}{ds}\right)^2-\frac{1}{3}\frac{d^2\rho}{ds^2},$$

where ρ is the radius of curvature at the point, and s the arc to that point from some fixed point. 86

4012. (Professor Townsend, F.R.S.)—Show that the internal surfaces of equilibrium of a uniform ellipsoidal shell of any thickness bounded by coaxal, but not similar surfaces, for the law of the inverse square of the distance, are in general two systems of coaxal and similar hyperboloids, of one and of two sheets respectively, separated by a cone asymptotic to them all; show also where they degenerate into two corresponding systems of hyperbolic cylinders, separated by a pair of planes asymptotic to them all. 56

4013. (G. M. Minchin, M.A.)—If $X=(1-\lambda^2x^2)(1-\mu^2x^2)$, prove that

$$\int_0^1\frac{x^2dx}{(1-\lambda^2x^2)X^{\frac{1}{2}}}+\int_0^1\frac{x^2dx}{(1-\mu^2x)X^{\frac{1}{2}}}+\int_0^1\frac{x^2dx}{X^{\frac{1}{2}}}=\frac{1}{\{(1-\lambda^2)(1-\mu^2)\}^{\frac{1}{2}}}.$$ 55

4055. (Dr. Hart.)—Find any number of cube numbers whose sum is a cube. 104

4066. (Professor Crofton, F.R.S.)—A straight line drawn at random crosses a circle; if two points are taken at random within the circle, show that the chance that they fall on opposite sides of the line is $\frac{128}{45\pi^2}$. 43

4067. (Professor Townsend, F.R.S.)—A uniform elastic beam, attached by its extremities to two fixed points A and B in the same horizonal line AB, and bent by its own weight W, is supported by a prop which keeps an intermediate point C in the same horizontal line with A and B; given all particulars, required the three pressures P, Q, R at the three points A, B, C..... 78

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4081. (S. Bills.)—Solve the two following equations:—
$$(x-6)^{\frac{1}{3}}+(x-3)^{\frac{1}{3}}+(x-2)^{\frac{1}{3}}+(x+9)^{\frac{1}{3}}=0 \quad \ldots\ldots (1),$$
$$(x^2-x+3)^{\frac{1}{3}}+(x^2-3x+1)^{\frac{1}{3}}+(x^2+3x-2)^{\frac{1}{3}}+(x^2-x-2)^{\frac{1}{3}}=10 \quad \ldots(2).$$ 38

4094. (Professor Townsend, F.R.S.)—Show that the cuspidal edge of any developable circumscribed to an umbilical quadric projects stereographically from an umbilic upon any parallel to its tangent plane into the evolute of the projection of its curve of contact with the quadric. 42

4101. (J. J. Walker, M.A.)—If O be the centre of the circle circumscribing the plane triangle ABC, and AO meet the base BC in A′, and O′ be taken harmonic conjugate to O with respect to AA′; if also O″ and O‴ be analogous points in reference to B and C respectively; prove (1) that the triangle O′O″O‴ circumscribes ABC; and (2) that
$$OO' : OO'' : OO''' = \cos^2 A : \cos^2 B : \cos^2 C \ldots\ldots 47$$

4127. (T. T. Wilkinson, F.R.A.S.)—If P be the intersection, and L, M, N the feet of the perpendiculars opposite to A, B, C respectively; prove that
$$AP^2 . BP^2 . CP^2 = 8R^3 . LP . MP . NP \ldots\ldots 110$$

4140. (Professor Townsend, F.R.S.)—A quartic curve being supposed determined on a ruled quadric by the intersection with it of any other quadric surface; prove that generally,

(*a*.) The problem, to construct, with generators of the quadric, a skew polygon of any even order inscribed to the quartic is either indeterminate or impossible.

(*b*.) Of the several generators of the quadric, two tetrads of opposite systems touch the quartic, and are equi-anharmonic with each other.

Regarding a binodal plane quartic curve as the projection of a twisted quartic in space; show immediately from the preceding (see SALMON's *Higher Plane Curves*, 2nd ed., arts. 282 and 270) that,

(*a*′.) The problem, to construct, with nodal chords of the curve, a polygon of any even order inscribed to it, is either indeterminate or impossible.

(*b*′.) Of the several nodal lines of the curve, two tetrads through opposite nodes are tangents to it, and are equi-anharmonic with each other. 23

4145. (J. J. Walker, M.A.)—Prove that the equation to the circle passing through a point $(x'\ y')$ and through the points of contact of tangents from it to the parabola $y^2-4ax=0$ is
$$a(x^2+y^2)-(2a^2+y'^2)x-y'(a-x')y+ax'(2a-x')=0 \ldots\ldots 48$$

4152. (Paul Serret.)—Une ellipse et l'un de ses cercles directeurs étant tracés, il existe une infinité de triangles simultanément sincrits au cercle et circonscrits à l'ellipse; le point de rencontre des hauteurs est le même pour tous ces triangles..... 70

4172. (Professor Wolstenholme.)—If two points fixed in a lamina slide upon two straight lines fixed in space (in one plane), and if one point be made to oscillate backwards and forwards so as to have always the same velocity, the ellipse described by any fixed point of the lamina will be described under acceleration which is fixed in direction. 27

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4175. (Dr. Ball, F.R.S.)—A free rigid body is acted upon by five forces; show how to move the body so that it shall not do work against nor receive energy from any one of the forces. 52

4179. (J. J. Walker, M.A.)—A fixed ray of light falls upon a plane mirror turning about an axis in its plane; show that the reflected rays will generate a ruled quadric surface. 36

4190. (C. Leudesdorf.)—A circular target is divided into three rings by drawing (1) a concentric circle within it, (2) a second concentric circle within the last; the radii of (1) and (2) being taken at random. If two arrows be shot so as both to strike the target, show that the chances that both lie within the same ring, that they lie in adjacent rings, and that they lie in non-adjacent rings are $\frac{1}{225}\frac{2}{8}$, $\frac{4}{150}$, $\frac{1}{90}\frac{3}{0}$. 22

4197. (Professor Wolstenholme, M.A.)—A triangular frame ABC without weight is kept in equilibrium by three forces at right angles to the sides whose lines of action meet in a point L. O is the centre of the circle ABC, LO is produced to L′ so that OL′ = LO; prove that the stresses at A, B, C will be perpendicular and proportional to L′A, L′B, L′C. 42

4199. (Professor Clifford, M.A.)—Three ternary quadrics U, V, W break up into linear factors 1, 1′; 2, 2′; 3, 3′ respectively. Prove that

$$\square\,(U, V, W) \equiv 123 \,.\, 1'2'3' + 1'23 \,.\, 12'3' + 12'3 \,.\, 1'23' + 123' \,.\, 1'2'3',$$

where $\square\,(U, V, W)$ is the coefficient of $6\lambda\mu\nu$ in the discriminant of $\lambda U + \mu V + \nu W$, and 123 means the determinant formed with the coefficients of the linear factors 1, 2, 3. Required developments and interpretations. 38

4201. (Rev. Dr. Booth, F.R.S.)—Prove (1) that the four points of contact of a twisted quadrilateral with a surface of the second order lie in a plane; and (2) that the tangent planes containing the sides of the quadrilateral meet in one point, viz., the pole of this plane. 27

4206. (R. Tucker, M.A.)—P is a point on an ellipse (foci S, H), and K is a point on the major axis such that PK = PS or PH; show (1) that PK touches a sextic; also (2) find the intersection of PK with the similar line for the point D (CD being conjugate to CP). 109

4209. (A. Martin.)—One end of a heavy uniform chain is suspended from a peg in the side of a vertical wall. If the other end be moved on a horizontal line through the peg, find the locus of the middle point of the chain. 61

4214. (E. B. Seitz.)—The first of two casks contains a gallons of wine, and the second contains b gallons of water. c gallons are drawn from the second cask, and then c gallons are drawn from the first cask and poured into the second, and the deficiency in the first supplied with c gallons of water. Required the quantity of wine in the second cask after n such operations as the one described above. 80

4215. (S. Watson.)—Let ABC be a triangle; O its centroid; D_1, E_1, F_1 the middle points of BC, CA, AB; D_2, E_2, F_2 the middle points

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of E_1F_1, F_1D_1, D_1E_1; and so on. Then if P be any point in the plane of the triangle, prove that

$$PD_n^2 + PE_n^2 + PF_n^2 = 3PO^2 + \frac{1}{3 \cdot 4^n}(BC^2 + CA^2 + AB^2). \quad \ldots\ldots 101$$

4227. (The Editor.)—Show that (1) the Boothian tangential equation of the evolute of the quadrantal hypocycloid is

$$a^2(\xi^2 - \eta^2)^2 = \xi^2 + \eta^2;$$

and (2) the equation of an involute of the hypocycloid is

$$a^2(\xi^2 - \eta^2)^2 = 4(\xi^2 + \eta^2)\{1 - c(\xi^2 + \eta^2)^{\frac{1}{4}}\}^2 \ldots\ldots 52$$

4231. (J. J. Sylvester, F.R.S.)—1. If two opposite angles of a jointed rhombus move in the same fixed circle, and one of the other angles in a second fixed circle passing through the centre of the first, prove that the fourth angle will move in a straight line.

2. Show that it is possible, by means of an apparatus consisting exclusively of right rods, compass joints, and pivots, to convert circular into linear, hyperbolic, elliptic, and parabolic motion; and, in general, to describe any curve of the form $x\phi(x^2, y^2) - \psi(x^2, y^2) = 0$, where ϕ, ψ are homogeneous forms of functions of any degree respectively in x^2, y^2.

NOTE.—This is the admirably simple solution (discovered within the last few years by the French geometer Peaucellier) of a problem that had generally been supposed to be as ineffectuable as the quadrature of the circle, and the assumed impossibility of which many mathematicians of the first order had been long labouring to demonstrate; viz., the rigorously exact commutation of circular with rectilinear motion by means of a combination consisting exclusively of rods, joints, and points. If the second circle does not pass through the centre of the first, the fourth angle describes a circle instead of a straight line ... 57

4233. (Professor Townsend, F.R.S.)—Show that every arc of a bifocal conic, described freely under the action of a central force emanating (in the proper direction) from either focus, is brachystochronous, for the velocity of its free description, under the action of an equal force emanating (in the proper direction) from the other focus. State also the general property of brachystochronous compared with free motion, of which the above is a particular case. 55

4234. (The Editor.)—1. Given the base of a triangle, and (α) the difference of the angles at the base, or (β) that the bisector of the vertical angle is parallel to a fixed straight line, or (γ) that the sides make equal angles with two fixed straight lines; prove that the locus of the vertex will be a rectangular hyperbola, having its centre at the middle point of the base, and its asymptotes parallel to the bisectors of the vertical angle.

2. Given two fixed tangents to a variable parabola, and a fixed point through which the axis always passes: prove that the locus of the focus will be a rectangular hyperbola, having its centre at the middle point of the line joining the fixed point to the intersection of the fixed lines, and its asymptotes parallel to the bisectors of the angle between those lines. 35

4235. (Rev. W. Roberts, M.A.)—Find the locus of points on the

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ellipses of a confocal system, at which the tangents cut off a constant area from the axes. A system (A) of curves being obtained by supposing the constant area to take different values, it is required to determine the system which cuts (A) orthogonally 24

4239. (Professor Wolstenholme, M.A.)—If

$$\frac{\sin(\alpha+\beta-\theta)}{\sin(\alpha+\beta)\cos^2\gamma} = \frac{\sin(\alpha+\gamma-\theta)}{\sin(\alpha+\gamma)\cos^2\beta},$$

β, γ being unequal and $< \frac{1}{2}\pi$, then will each of these be equal to

$$\frac{\sin(\beta+\gamma-\theta)}{\sin(\beta+\gamma)\cos^2\alpha};$$

and $\cot\theta = \dfrac{\sin^2(\alpha+\beta+\gamma)+\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)}{\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)}$. 94

4242. (S. Roberts, M.A.)—The two ovals of a Cartesian meet the axis in four real points A, B, C, D, and in the same succession as the letters. The axial diameters may therefore be grouped in three pairs [AB, CD], [AC, BD], [AD, BC]. If three ellipses are drawn with their principal semi-diameters equal to the three pairs of diameters of the Cartesian, then periphery of outer oval = ½ sum of peripheries of ellipses on [AC, BD], [AB, CD], periphery of inner oval = ½ difference of same, and the arcs of the ovals intercepted by the axis and the apices, points of contact of tangents from the exterior focus, are expressible in terms of arcs of the third ellipse and the peripheries of the first ellipses. In the case of a nodal limaçon, the loop periphery = ½ (periphery of ellipse whose principal semi-diameters are equal to the diameters of the loops ± four times the difference of the loop diameters). 91

4263. (G. M. Minchin, M.A.)—A rigid body is acted on by any system of forces. If from any point on Poinsot's Central Axis a right line be drawn perpendicular to this axis, show that the principal axes of moments at the various points along this right line trace out a hyperbolic paraboloid. 29

4265. (Sir James Cockle, F.R.S.)—Boole (*Differential Equations*, 2nd ed., p. 350) says that the method of Charpit supposes that q can be expressed as a function of x, y, z, p. He subsequently (p. 351) notes that it presupposes the existence of two algebraic equations between x, y, z, p, q. Explain this..... 28

4268. (Professor Wolstenholme, M.A.)—If A, B, C be three points on a parabola; A′B′C′ the triangle formed by the tangents at A, B, C; and a, b, c the centres of the circles BCA′, CAB′, ABC′; prove that the circle abc will pass through the focus.... 72

4270. (Professor Crofton, F.R.S.)—1. The resultant of a system of forces acting in the same sense along the sides of a portion AF of a polygon, and whose magnitudes are respectively m.AB, n.BC, p.CD, q.DE, r.EF, is, in magnitude and direction (not position), r.GF; where G is the centre of gravity of a system of weights represented by $m, n-m, p-n, q-p, r-q$; placed respectively at A, B, C, D, E. Some of these may of course be *negative* weights. Likewise for any number of forces.

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2. Tangential stress acts along any arc AB of a curve, of intensity always proportional to the distance from A measured along the curve. Show that the resultant is in magnitude and direction, $i \cdot GB$; where i is the intensity of the stress at B, and G is the centre of gravity of the arc AB. 25

4272. (C. W. Merrifield, F.R.S.)—From the corners of a triangle ABC lines are drawn through a point O, meeting the opposite sides in DEF. Through A a line B_1C_1 is drawn parallel and equal to BC; and OB_1, OC_1 are drawn to meet BC in B_2C_2. If C_3A_2, A_3B_3 are analogous segments on CA, AB respectively; prove (1) that

$$AF \cdot BD \cdot CE = BF \cdot CD \cdot AE = B_2C_2, C_3A_2 \cdot A_3B_3,$$

and (2) find a corresponding theorem in regard to lines drawn through the angles parallel to any transversal DEF. 70

4275. (W. H. H. Hudson, M.A.)—If S, H be the poles of a Lemniscate, P any point on it, C the middle point of SH, and if PH′, PS′ be taken on PS, PH, equal respectively to PH, PS; prove that the normal at P bisects S′H′. 46

4276. (J. J. Walker, M.A.)—If h_1, h_2, h_3 are the depths of the corners A, B, C of a plane triangle below the surface of water; prove that the depth (k) of the centre of pressure (K) is given by

$$k = \frac{h_1^2 + h_2^2 + h_3^2 + h_2h_3 + h_3h_1 + h_1h_2}{2(h_1 + h_2 + h_3)}. \quad \ldots\ldots 90$$

4279. (A. W. Panton, M.A.)—Prove that the sum of the areas of the two ovals of a Cartesian is equal to twice the area of the circle whose centre is the triple focus, and which passes through the points of contact of the double tangent. 89

4282. (F. H. Dinnis, M.A.)—ABCD is a quadrilateral inscribed in a circle. A circle round DCE (E being the intersection of the diagonals) meets BC, AD in G and H respectively, and GH intersects CD in L. A circle round ABE meets BC, AD in F and K respectively, and FK intersects AB in M. Prove that LM is parallel to L′M′, where L′, M′ are points similarly determined by circles round AED, BEC. 73

4284. (J. L. McKenzie.)—If l, m, n be the lines bisecting the angles of the triangle ABC, and terminating in the opposite sides, and Δ the area; prove that

$$\Delta = \tfrac{1}{2}l(b+c)\sin\frac{A}{2} = \tfrac{1}{2}la\cos\tfrac{1}{2}(B-C) = \frac{lmn(b+c)(c+a)(a+b)}{4abc(a+b+c)}. \quad 54$$

4291. (Professor Tait.)—Upon what element of a planet's orbit, supposed nearly circular, will the attraction of masses external to the solar system produce the greatest effect. 60

4298. (J. W. L. Glaisher, B.A.)—With four given straight lines to form a quadrilateral inscribable in a circle. 29

4309. (R. Tucker, M.A.)—Solve the equations

$$-x^2 + y^2 + z^2 - xy + yz - xz = a = 13 \quad \ldots\ldots (1),$$
$$x^2 - y^2 + z^2 - xy - yz + xz = b = 1 \quad \ldots\ldots (2),$$
$$x^2 + y^2 - z^2 + xy - yz - xz = c = -11 \quad \ldots\ldots (3). \quad 40$$

4311. (T. T. Wilkinson, F.R.A.S.)—Find the point in the side of a

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given rectangular billiard table, adjacent to the pocket, from which if a ball be struck so as to hit each of the other three sides in succession, and enter the pocket, it may then have run a length equal to the perimeter of the table. 98

4317. (The Editor.)—Prove that the sum of the squares of the radius of curvature at any point of a cycloid, and of the arc from that point to the vertex of the curve, is constant; and extend this property to epicycloids. 89

4320. (J. J. Sylvester, F.R.S.)—If two points, taken equi-distant from one of the angles of a jointed equilateral rhomb on the adjacent sides, move in the same circle, and the opposite angle in a circle passing through the centre of the first; prove that the angle first named will move in a certain third circle. 58, 111

4322. (Professor Crofton, F R.S.)—A heavy uniform beam AB is supported on two props at the points C, D; prove the following construction for finding the two points were the bending moment vanishes. With C, D as centres draw two circles passing respectively through A and B; draw any circle cutting both these circles orthogonally; it will cut the beam AB in the two points of inflexion required. The beam might therefore be jointed at these points. If the circle does not meet AB, there is no such point, and the beam is convex upwards throughout its whole length..... 49

4324. (The Editor.)—1. A constant angle α revolves round its vertex, which is fixed at a focus of a conic whose excentricity is e, and semi-parameter p; and a tangent is drawn at the point where the conic is cut by one of the sides of the angle; show that the locus of the point in which this tangent meets the other side of the angle is a conic whose excentricity is $e \sec \alpha$, and semi-parameter $p \sec \alpha$.

2. The base of a triangle, and the sum or the difference of the sides, are given; find the locus of the point in which the bisector of the exterior or the interior vertical angle meets a straight line making a given angle with one of the sides, and drawn through one end of the base. 66

4326. (Rev. Dr. Booth, F.R.S.)—Let a, b, c, d be the sides, and S the area of a quadrilateral circumscribed about one circle, and inscribed in another of radius R; prove that

$S^2 = abcd$, and $16 R^2 = a^2 + b^2 + c^2 + d^2 + S^2 (a^{-2} + b^{-2} + c^{-2} + d^{-2})$. 73

4328. (Professor Wolstenholme.)—Prove that the tangent to the evolute of a parabola at the point where it cuts the parabola is also a normal to the evolute. 79

4332. (H. S. Monck.)—The form of the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ implies that if on the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 0$ perpendiculars be drawn from any point of the circumference, their product will be constant. The equation of the ellipse seems to imply a corresponding property with regard to the curve and the imaginary asymptotes, viz., $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and

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$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$. Draw a pair of lines which shall possess the property in question. 105

4335. (Christine Ladd.)—From each vertex of a quadrilateral draw perpendiculars upon two sides and a diagonal, and show that the triangles formed by joining the feet of each set of perpendiculars are similar. 62

4339. (Professor Townsend, F.R.S.)—Show that the two principal axes at the centre of gravity, in the plane of a uniform triangular lamina, are the two rectangular conjugates of the involution determined by the three pairs of lines through the point which bisect internally and externally the three sides of the triangle. 74

4345. (Rev. Dr. Booth, F.R.S.)—Prove that the semi-diameter of an ellipse along the excentric radius of the circle drawn on the major axis, is equal to the central perpendicular drawn on the tangent to the ellipse through the point where the ordinate meets the ellipse..... 110

4349. (B. Williamson, M.A.)—Give an elementary proof of Landen's theorem, that the length of any arc of an hyperbola can be expressed by means of the arcs of two ellipses. 103

4350. (Colonel Clarke, C.B., F.R.S.)—Three points are taken at random in a given triangle: prove that the probability (p) that they enclose a given point whose triangular coordinates are (x, y, z), is $p = \mathrm{F}(x, y, z) + \mathrm{F}(y, z, x) + \mathrm{F}(z, x, y)$,

where $\mathrm{F}(x, y, z) = 2xyz(1-6x^2) + (6y^2z^2 - 24y^3z^3)\log\left(1+\frac{x}{yz}\right)$. 63

4351. (A. W. Panton, M.A.)—If the roots of the equation

$$a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{1.2}a_2x^{n-2} + \ldots = 0$$

be in arithmetical progression, show that they are obtained from the expression

$$-\frac{a_1}{a_0} \pm \frac{p}{a_0}\left(\frac{3(a_1^2 - a_0a_2)}{n+1}\right)^{\frac{1}{2}}$$

by giving p all the values 1, 3, 5, ... $(n-1)$ when n is even, and all the values 0, 2, 4 ... $(n-1)$ when n is odd. 97

4358. (J. L. McKenzie.)—1. A given ellipse U is one of a system of concentric, similar, and similarly situated ellipses. A line is drawn touching any other ellipse V of the system; and the perpendicular distance of the tangent from the centre is a mean proportional between the semi-axis major of V and the semi-axis minor of U. Show that the envelope of the tangent is the first negative pedal of U, but rotated through a right angle about its centre.

2. If the duplicate ratio of the perpendicular on the tangent to the linear unit, be equal to the ratio of linear similarity of V and U, show that the envelope of the tangent is the first negative pedal of the polar. 65

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4366. (Professor Crofton, F.R.S.)—Show that a normal stress, acting all round a closed convex curve, of an intensity everywhere inversely proportional to the radius of curvature, makes equilibrium. Find the resultant of the stress on any arc of the curve. 99

4370. (Professor Townsend, M.A.)—A rigid body, in unconstrained equilibrium in free space, being supposed set in motion by a single impulsive force, show directly that

(*a*.) If the central axis of initial rotation be a principal axis of the body, the initial motion is one of pure rotation; and, conversely, if the initial motion be one of pure rotation, the central axis of initial rotation is a principal axis of the body.

(*b*.) For different initial axes of pure rotation having a common direction in the body, and lying consequently in a plane passing through its centre of gravity, the locus of the corresponding centres of percussion is a right line passing through the centre of gravity and lying in the plane of the axes. 107

4371. (The Editor.)—Three of the vertices of a quadrilateral move round the curve of an ellipse, two of the sides pass through the foci, and the other two are tangents to the ellipse; show that the locus of the fourth vertex is an ellipse having its centre and axes on those of the given ellipse, its minor semi-axis equal to a, and its major semi-axis equal to $2a^2b^{-1}-b$, where a and b are the semi-axes of the given ellipse..... 95

4372. (A. L. Pirie, M.A.)—ABC is a triangle inscribed in a circle; from any point O on the circumference perpendiculars are drawn to OA, OB, OC, meeting BC, CA, AB, in D, E, F: prove that D, E, F lie on a straight line which passes through the centre of the circle..... 93

4375. (J. J. Walker, M.A.)—If α, β, γ are the angles which the bisectors of the sides of a plane triangle drawn from A, B, C respectively make with the side BC, prove that

$$\tan\alpha\,(\tan\gamma\tan B-\tan\beta\tan C)-(\tan\beta+\tan\gamma)\tan B\tan C + \tan\beta\tan\gamma\,(\tan B+\tan C)=0$$ 106

4383. (W. B. Davis, B.A.)—If the sides of a quadrilateral be produced, and four circles are inscribed, prove that the centres of these circles lie on the circumference of another circle. 83

4389. (Rev. Dr. Booth, F.R.S.)—A triangle circumscribes a parabola; prove that (1) the square of the product of the three sides of the triangle is equal to the product of the six tangents drawn to the curve from the three angles of the triangle; and (2) the product of the three focal vectors drawn to the angles of the triangle is equal to the product of the three focal vectors drawn to the points of contact..... 102

4391. (Professor Wolstenholme, M.A.)—Prove that

$$\begin{vmatrix} a, & 1, & 0, & 0, & 0, \ldots\ldots & 0 \\ 1, & a, & 1, & 0, & 0, \ldots\ldots & 0 \\ 0, & 1, & a, & 1, & 0, \ldots\ldots & 0 \\ \multicolumn{6}{c}{\ldots\ldots\ldots\ldots} \\ 0, & 0, & \ldots\ldots & 1, & a, & 1 \\ 0, & 0, & \ldots\ldots & 0, & 1, & a \end{vmatrix} \equiv \frac{\sin(n+1)\theta}{\sin\theta}, \text{ where } 2\cos\theta = a \text{; and also}$$

$$\equiv a^n-(n-1)a^{n-2}+\frac{(n-2)(n-3)}{\lfloor 2}a^{n-4}-\ldots$$ 83

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4392. (S. Roberts, M.A.)—If N_p denotes the number of terms in a symmetrical determinant of p rows and columns; show that the successive numbers are given by the equation

$$N_k - N_{k-1} - (k-1) N_{k-2} + \tfrac{1}{2}(k-1)(k-2)\{N_{k-3} + (k-3) N_{k-4}\} = 0,$$

k being positive and N_0 being taken equal to unity. 81

4395. (J. W. L. Glaisher, B.A.)—Prove that, n being a positive integer,

$$\int_0^1 \int_0^\pi x^{-\rho x \cos\theta} \cos(\rho x \log x \sin\theta + n\theta)\, dx\, d\theta = \frac{\pi \rho^n}{(n+1)^{n+1}}. \quad \ldots\ldots \quad 85$$

4406. (J. L. McKenzie.)—Find the least triangle whose sides are integers in arithmetical progression, and the perpendiculars on the sides from the opposite angles integers in harmonical progression. 108

4409. (Geometricus.)—The base and vertical angle of a triangle are given, and equilateral triangles are constructed on the two sides; find the locus of the middle point of the line joining the vertices of the equilateral triangles. 109

CORRIGENDA.

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In the solution of Question 4168, on p. 53, the expression in Art. 1 should be replaced by

$$e^{-2\int P dx} - \frac{d}{dx}\left\{\frac{e^{-2\int P dx}}{y - P}\right\}.$$

This oversight of the author's does not affect the conclusion.

Page 74, line 9 from bottom, *insert* $\frac{2}{l}$ *before* $(U + U_1)$.

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Page 55, line 10, *after* y, *add* $=$.
Page 73, line 21, *for* DENNIS, *read* DINNIS.

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MATHEMATICS

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THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2600. (Proposed by the EDITOR.)—Show that (1) the area of a parabolic section of a right circular cone whose slant height is unity and semi-vertical angle α, made by a plane at a distance z from the vertex along the slant side, is $\frac{8}{3}z^{\frac{3}{2}}(1-z)^{\frac{3}{2}}\sin\alpha$; (2) the greatest parabola of the series has an area of $\cos\frac{1}{3}\pi\sin\alpha$ (or ·866 $\sin\alpha$), and its vertex at a distance of one-fourth of the slant height from the vertex of the cone; (3) if the cutting planes be uniformly distributed, the mean area of all these parabolas is $\frac{1}{6}\pi\sin\alpha$ (or ·524 $\sin\alpha$); (4) if x be the perpendicular distance of the cutting plane from the vertex of the cone, and θ the inclination of this perpendicular to the axis of the cone, the area of a complete elliptic section is $\pi x^2\sin^2\alpha\cos\alpha\sec^{\frac{1}{2}}(\theta+\alpha)\sec^{\frac{1}{2}}(\theta-\alpha)$; (5) if $\alpha<\frac{1}{11}\pi$, and the section reaches to the base of the cone, the ellipse is a maximum when θ is the greater of the two acute-angled roots of the equation $\sin 2\theta = 2\sin 2\alpha$, and a minimum when θ is the less of these roots; but if $\alpha =$ or $>\frac{1}{11}\pi$ the ellipse increases continuously from coincidence with the edge to coincidence with the base of the cone; (6) the eccentricity of any section is $\sec\alpha\sin\theta$, which is a maximum when the cutting plane is parallel to the axis; (7) if the cutting planes be equably distributed in space, the mean eccentricities of the ellipses, the hyperbolas, and all the sections of the cone, are respectively

$$\frac{2\cos^3\alpha+2\tan\alpha(2-3\sin\alpha+\sin^3\alpha)}{3\cos\alpha\cos 2\alpha+3(\frac{1}{2}\pi-\alpha)\sin\alpha},\quad \frac{2\tan\alpha(5+\cos 2\alpha)}{3(2\alpha+\sin 2\alpha)},$$

$$\frac{2\cos^3\alpha+2\tan\alpha(2+3\sin\alpha-\sin^3\alpha)}{3\cos\alpha+3(\frac{1}{2}\pi+\alpha)\sin\alpha},$$

which, when the cone is right-angled at the vertex, become

$$\frac{16}{3\pi}(\sqrt{2}-1) = \cdot 703,\quad \frac{20}{3(2+\pi)} = 1\cdot 297,\quad \frac{24+16\sqrt{2}}{12+9\pi} = 1\cdot 158;$$

(8) the respective probabilities that a random plane will cut a complete ellipse, an ellipse of any kind, or an hyperbola, are

$$\frac{\cos\alpha-(\frac{1}{2}\pi-\alpha)\sin\alpha}{\cos\alpha+(\frac{1}{2}\pi+\alpha)\sin\alpha},\quad \frac{\cos\alpha\cos 2\alpha+(\frac{1}{2}\pi-\alpha)\sin\alpha}{\cos\alpha+(\frac{1}{2}\pi+\alpha)\sin\alpha},\quad \frac{2\cos\alpha\sin^2\alpha+2\alpha\sin\alpha}{\cos\alpha+(\frac{1}{2}\pi+\alpha)\sin\alpha}$$

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which, for a right-angled cone, become

$$\frac{4-\pi}{4+3\pi} = \cdot 064, \quad \frac{\pi}{4+3\pi} = \cdot 234, \quad \frac{4+2\pi}{4+3\pi} = \cdot 766;$$

(9) the mean area of all the complete ellipses is

$$\frac{\frac{2}{3}\pi \sin^2\alpha \cos\alpha}{\cos\alpha - (\frac{1}{2}\pi - \alpha)\sin\alpha} \int_0^{\frac{1}{2}\pi-\alpha} \frac{\cos^{\frac{3}{2}}(\theta+\alpha)}{\cos^{\frac{3}{2}}(\theta-\alpha)} \sin\theta\, d\theta;$$

(10) in a right-angled cone, the mean area of the complete ellipses is

$$\frac{(2\pi)^{\frac{3}{2}}}{12(4-\pi)} \left(\left\{\Gamma(\tfrac{1}{4})\right\}^2 + 12\left\{\Gamma(\tfrac{3}{4})\right\}^2 \right) - \frac{2\pi}{4-\pi} = \cdot 26409.$$

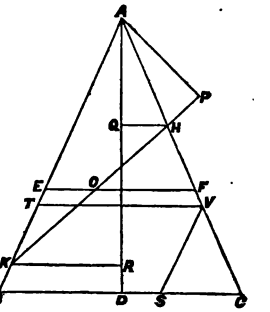
Solution by the PROPOSER.

1. Let the figure represent an axial section of the cone perpendicular to the cutting plane; then, if VS be the axis of the parabola, and VT parallel to BC, we have

$$VS = VC = 1-z, \quad BS = TV = 2z\sin\alpha,$$
$$SC = 2(1-z)\sin\alpha;$$

$\therefore$ Area of parabola $= \frac{4}{3} VS\,(BS \cdot SC)^{\frac{1}{2}}$

$$= \tfrac{8}{3} z^{\frac{1}{2}} (1-z)^{\frac{1}{2}} \sin\alpha.$$



2. Let $f(z) = z(1-z)^3$,

then $f'(z) = (1-z)^2(1-4z)$,

and $f''(z) = -6(1-z)(1-2z)$;

hence the area of the parabola is a maximum ($= \frac{1}{4}\sqrt{3}\sin\alpha$) when $z = \frac{1}{4}$.

3. All the different parabolic sections will be taken into account if we consider all those made by planes having a common normal in the plane ABC, and of course also perpendicular to AB; and the cutting planes will be uniformly distributed if they cut off equal increments along this normal, or what is the same thing, along the side AC of the cone; thus the average of the areas of the parabolas is

$$\tfrac{8}{3}\sin\alpha \int_0^1 z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\, dz = \tfrac{8}{3}\sin\alpha \frac{\Gamma(\frac{3}{2})\cdot\Gamma(\frac{3}{2})}{\Gamma(4)}$$
$$= \tfrac{8}{3}\sin\alpha \cdot \tfrac{1}{16}\left\{\Gamma(\tfrac{1}{2})\right\}^2 = \tfrac{1}{2}\pi\sin\alpha.$$

The value of this definite integral may be otherwise obtained, without using the Gamma-function, by putting $z = \sin^2\phi$, when it becomes

$$2\int_0^{\frac{1}{2}\pi} \sin^2\phi\cos^4\phi\, d\phi = 2\int_0^{\frac{1}{2}\pi} (\cos^4\phi - \cos^6\phi)\, d\phi$$
$$= 2\left(\frac{1.3}{2.4} - \frac{1.3.5}{2.4.6}\right)\pi = \frac{\pi}{16}.$$

Or again, the value of the integral may be more simply found as follows:—

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$$\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\,dx = \int_0^1 (1-x)^{\frac{1}{2}}x^{\frac{1}{2}}\,dx$$

$$= \int_0^1 \tfrac{1}{2}\left\{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} + x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\right\}dx = \tfrac{1}{2}\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\,dx$$

$$= \tfrac{1}{4}\,(\text{Area of circle of unit-diameter}) = \tfrac{1}{16}\pi.$$

4. Let HK be the axis of the ellipse, O its centre, and AP $(=x)$ the perpendicular on the cutting plane from the vertex A of the cone; then, drawing HQ, EOF, KR parallel to BC, we have

$$\text{major semi-axis} = \text{OH} = \tfrac{1}{2}x\left\{\tan(\theta+\alpha)-\tan(\theta-\alpha)\right\}$$
$$= x\sin\alpha\cos\alpha\sec(\theta+\alpha)\sec(\theta-\alpha),$$
$$\text{minor semi-axis} = (\text{EO}\,.\,\text{OF})^{\frac{1}{2}} = (\text{HQ}\,.\,\text{KR})^{\frac{1}{2}}$$
$$= x\sin\alpha\sec^{\frac{1}{2}}(\theta+\alpha)\sec^{\frac{1}{2}}(\theta-\alpha);$$

hence the Area of the Ellipse $= \pi x^2\sin^2\alpha\cos\alpha\sec^{\frac{3}{2}}(\theta+\alpha)\sec^{\frac{3}{2}}(\theta-\alpha)$.

5. When K is at B and therefore $x = \cos(\theta+\alpha)$, the area is

$$\pi\sin^2\alpha\cos\alpha\cos^{\frac{3}{2}}(\theta+\alpha)\sec^{\frac{3}{2}}(\theta-\alpha).$$

Let $f(\theta) = \cos(\theta+\alpha)\sec^3(\theta-\alpha)$, then $f'(\theta) = \sec^4(\theta-\alpha)(\sin 2\theta - 2\sin 2\alpha)$. When, therefore, $\alpha < \frac{1}{12}\pi$ or $\sin 2\alpha < 1$, we see that $f'(\theta)$ is *negative* as θ increases up to the first root of the equation $\sin 2\theta = 2\sin 2\alpha$; *zero* at that root, thus indicating that $f(\theta)$ has there decreased to a minimum; thence *positive* up to the second root of the equation, showing that $f(\theta)$ has there increased to a maximum; then it passes a second time through zero, and again becomes negative. It is obvious that $f'(\theta)$ can never vanish when $\alpha > \frac{1}{12}\pi$, since then the roots of the equation $\sin 2\theta = 2\sin 2\alpha$ are imaginary. Moreover when $\alpha =$ or $> \frac{1}{12}\pi$, then $2\sin 2\alpha =$ or > 1 and $f'(\theta)$ can never be positive, therefore $f(\theta)$ decreases continuously as θ increases.

6. The value $\sec\alpha\sin\theta$ for the eccentricity may be readily deduced from the values of the semi-axés given in Art. 4; and the same has been otherwise obtained, for any section, in the solutions of Quest. 3924, on pp. 88—90 of Vol. XVIII. of the *Reprint*. The eccentricity has obviously its maximum value $\sec\alpha$ when $\sin\theta = 1$ or $\theta = \frac{1}{2}\pi$, that is, when the cutting plane is parallel to the axis of the cone.

7. The cutting planes will be uniformly distributed in space, provided the points in which the normals on these planes from the vertex of the cone meet the sphere of unit radius around this vertex as centre are equably distributed over its surface. Thus the number of cutting planes within an increment $d\theta$ of the inclination of the normal to the axis of the cone is proportional to the area of a zone of the unit-sphere of breadth $\sin\theta\,d\theta$, and the surfaces of zones vary as their breadths; hence $\sin\theta\,d\theta$ is the measure of the number of planes corresponding to the increment $d\theta$.

The *complete* ellipses have a normal range of $\cos(\theta+\alpha)$, and an amplitude of normal inclination to the axis of the cone from 0 to $\frac{1}{2}\pi-\alpha$; hence the measure of the number of *complete* ellipses is

$$\int_0^{\frac{1}{2}\pi-\alpha}\cos(\theta+\alpha)\sin\theta\,d\theta = \tfrac{1}{2}\Big[\sin\theta\sin(\theta+\alpha)-\theta\sin\alpha\Big]_{\theta=0}^{\theta=\frac{1}{2}\pi-\alpha}$$
$$= \tfrac{1}{2}\cos\alpha - \tfrac{1}{2}(\tfrac{1}{2}\pi-\alpha)\sin\alpha \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (\beta).$$

Similarly, the measure of the total number of ellipses of any kind is

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$$\int_0^{\frac{1}{2}\pi-\alpha} \cos(\theta-\alpha)\sin\theta\,d\theta = \tfrac{1}{2}\left[\sin\theta\sin(\theta-\alpha)+\theta\sin\alpha\right]_{\theta=0}^{\theta=\frac{1}{2}\pi-\alpha}$$
$$= \tfrac{1}{2}\cos\alpha\cos 2\alpha + \tfrac{1}{2}(\tfrac{1}{2}\pi-\alpha)\sin\alpha \ldots\ldots\ldots (\gamma).$$

The hyperbolas have a normal range of $\cos(\theta-\alpha)$ and an amplitude from $\frac{1}{2}\pi-\alpha$ to $\frac{1}{2}\pi+\alpha$; hence the number of hyperbolas is

$$\int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi+\alpha} \cos(\theta-\alpha)\sin\theta\,d\theta = \tfrac{1}{2}\left[\sin\theta\sin(\theta-\alpha)+\theta\sin\alpha\right]_{\theta=\frac{1}{2}\pi-\alpha}^{\theta=\frac{1}{2}\pi+\alpha}$$
$$= \cos\alpha\sin^2\alpha + \alpha\sin\alpha \ldots\ldots\ldots (\delta).$$

The measure of the total number of sections is, therefore,

$$(\gamma)+(\delta) = \tfrac{1}{2}\cos\alpha + \tfrac{1}{2}(\tfrac{1}{2}\pi+\alpha)\sin\alpha \ldots\ldots\ldots (\epsilon).$$

The sum of the eccentricities of all the ellipses is

$$\int_0^{\frac{1}{2}\pi-\alpha} \sec\alpha\sin^2\theta\cos(\theta-\alpha)\,d\theta = \left[\tfrac{1}{3}\sin^3\theta - \tan\alpha(\cos\theta-\tfrac{1}{3}\cos^3\theta)\right]_{\theta=0}^{\theta=\frac{1}{2}\pi-\alpha}$$
$$= \tfrac{1}{3}\cos^3\alpha + \tfrac{1}{3}\tan\alpha(2-3\sin\alpha+\sin^3\alpha) \ldots (\zeta).$$

The sum of the eccentricities of all the hyperbolas is

$$\left[\tfrac{1}{3}\sin^3\theta - \tan\alpha(\cos\theta-\tfrac{1}{3}\cos^3\theta)\right]_{\theta=\frac{1}{2}\pi-\alpha}^{\theta=\frac{1}{2}\pi+\alpha} = \tfrac{2}{3}\sin\alpha\tan\alpha(3-\sin^2\alpha) \ldots\ldots (\eta).$$

The sum of the eccentricities of all the sections of the cone is

$$(\zeta)+(\eta) = \tfrac{1}{3}\cos^2\alpha + \tfrac{1}{3}\tan\alpha(2+3\sin\alpha-\sin^3\alpha) \ldots\ldots\ldots (\kappa).$$

The mean eccentricities of all the ellipses of the hyperbolas, and of all the sections are $\frac{(\zeta)}{(\gamma)}$, $\frac{(\eta)}{(\delta)}$, $\frac{(\kappa)}{(\epsilon)}$; and these lead to the results given in the Question.

8. The respective probabilities that a plane drawn at random will cut a complete ellipse, an ellipse of any kind, or an hyperbola, are $\frac{(\beta)}{(\epsilon)}$, $\frac{(\gamma)}{(\epsilon)}$, $\frac{(\delta)}{(\epsilon)}$; and these lead at once to the results stated in the Question.

9. The sum of the areas of all the complete ellipses is

$$\int_0^{\frac{1}{2}\pi-\alpha}\int_0^{\cos(\theta+\alpha)} \pi x^2\sin^2\alpha\cos\alpha\sec^{\frac{1}{2}}(\theta+\alpha)\sec^{\frac{1}{2}}(\theta-\alpha)\sin\theta\,d\theta\,dx.$$

Integrating with respect to x, and dividing the result by the number (β) of complete ellipses, we obtain the mean area stated in the Question.

10. When the cone is right-angled, the mean area becomes

$$\frac{4\pi}{12-3\pi}\int_0^{\frac{1}{2}\pi}\tan^{\frac{1}{2}}(\tfrac{1}{4}\pi-\theta)\sin\theta\,d\theta = \frac{\frac{4}{3}\pi}{4-\pi}\int_0^{\frac{1}{2}\pi}\tan^{\frac{1}{2}}\theta\sin(\tfrac{1}{4}\pi-\theta)\,d\theta \ldots (\lambda).$$

Putting Σ for the integral in (λ), and taking the *second* form, we have

$$\Sigma = \frac{1}{\sqrt{2}}\int_0^{\frac{1}{2}\pi}(\cos\theta-\sin\theta)\tan^{\frac{1}{2}}\theta\,d\theta = \frac{1}{\sqrt{2}}\int_0^{\frac{1}{2}\pi}(\sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta - \sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta)\,d\theta$$
$$\ldots\ldots (\mu),$$

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By the aid of the two following formulæ of reduction,

$$\int \sin^m\theta\cos^n\theta\,d\theta = -\frac{\sin^{m+1}\theta\cos^{n+1}\theta}{n+1} + \frac{m+n+2}{n+1}\int \sin^m\theta\cos^{n+2}\theta\,d\theta,$$

$$\int \sin^m\theta\cos^n\theta\,d\theta = -\frac{\sin^{m-1}\theta\cos^{n+1}\theta}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}\theta\cos^{n+2}\theta\,d\theta;$$

using the first for the first term of (μ), and the second for the second term, and then applying the formulæ

$$\int_0^{\frac{1}{2}\pi}\sin^n\theta\,d\theta = \frac{\Gamma(\frac{1}{2})\,\Gamma\{\frac{1}{2}(n+1)\}}{2\Gamma\{\frac{1}{2}(n+2)\}} \quad \text{(Todhunter's } \textit{Integral Calculus}\text{, Art. 272),}$$

$$\Gamma(1+n) = n\Gamma(n), \quad \Gamma(n)\,\Gamma(1-n) = \pi\operatorname{cosec} n\pi, \quad \Gamma(\tfrac{1}{2}) = \pi^{\frac{1}{2}},$$

we have

$$\mathfrak{A} = -\sqrt{2}\left(\sin^{\frac{1}{2}}\theta\cos^{\frac{1}{2}}\theta + \sin^{\frac{3}{2}}\theta\cos^{-\frac{1}{2}}\theta\right)_{\theta=0}^{\theta=\frac{1}{2}\pi} + \tfrac{3}{2}\int_0^{\frac{1}{2}\pi}(\sin^{\frac{1}{2}}2\theta + \sin^{\frac{3}{2}}2\theta)\,d\theta$$

$$= -\tfrac{3}{2} + \tfrac{3}{4}\int_0^{\frac{1}{2}\pi}(\sin^{\frac{1}{2}}\theta + \sin^{\frac{3}{2}}\theta)\,d\theta = -\tfrac{3}{2} + \tfrac{3}{8}\Gamma(\tfrac{1}{4})\left\{\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{7}{4})} + \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\right\}$$

$$= -\tfrac{3}{2} + \frac{1}{8(2\pi)^{\frac{1}{2}}}\left[\{\Gamma(\tfrac{1}{4})\}^2 + 12\{\Gamma(\tfrac{3}{4})\}^2\right].$$

Referring to De Morgan's *Calculus*, p. 590, for the logarithms of the Gamma-functions, the calculation of the values of $\mathfrak{A}$, and thence of the mean area of the ellipses, will be as follows:—

$$\{\Gamma(\tfrac{1}{4})\}^2 + 12\{\Gamma(\tfrac{3}{4})\}^2 = \log^{-1}(1{\cdot}11876215) + 12\log^{-1}(0{\cdot}17656759)$$
$$= 13{\cdot}145046 + 12(1{\cdot}501646) = 31{\cdot}164798;$$

$\log 31{\cdot}164798$	$= 1{\cdot}4936643$	$\log(\mathfrak{A} = {\cdot}05412)$	$= \bar{2}{\cdot}7333578$
$10 - \frac{1}{2}\log\pi$	$= 9{\cdot}7514251$	$\log(\pi = 3{\cdot}14159)$	$= 0{\cdot}4971499$
$10 - \frac{1}{2}\log 128$	$= 8{\cdot}9463950$	$2\log 2$	$= 0{\cdot}6020600$
		$10 - \log(12 - 3\pi = 2{\cdot}57523)$	$= 9{\cdot}5891839$
$\log 1{\cdot}55412$	$= 0{\cdot}1914844$		
		$\log(\text{Av.} = {\cdot}26409)$	$= \bar{1}{\cdot}4217516$

The value of $\mathfrak{A}$ may be otherwise found, without using the Gamma-function, by putting $\sin\theta = \sin^2\frac{1}{2}\phi$, and developing into a series; and we thus obtain $\int_0^{\frac{1}{2}\pi}(\sin^{\frac{1}{2}}\theta + \sin^{\frac{3}{2}}\theta)\,d\theta = \int_0^{\frac{1}{2}\pi}\sin^{\frac{1}{2}}\theta(1+\sin\theta)\,d\theta$

$$= \int_0^{\pi}\sin^2\tfrac{1}{2}\phi\,(1+\sin^2\tfrac{1}{2}\phi)^{\frac{1}{2}}\,d\phi = \frac{\sqrt{3}}{2\sqrt{2}}\int_0^{\pi}(1-\cos\phi)(1-\tfrac{1}{3}\cos\phi)^{\frac{1}{2}}\,d\phi$$

$$= \frac{\sqrt{3}}{2\sqrt{2}}\int_0^{\pi}d\phi\,(1-\cos\phi)\left\{1 - \left(\frac{\cos\phi}{6}\right) - \frac{1}{2}\left(\frac{\cos\phi}{6}\right)^2 - \frac{1.1.3}{1.2.3}\left(\frac{\cos\phi}{6}\right)^3\right.$$
$$\left. - \frac{1.1.3.5}{1.2.3.4}\left(\frac{\cos\phi}{6}\right)^4 - \ldots\ldots - \frac{3.5\ldots(2r-3)}{\Pi(r)}\left(\frac{\cos\phi}{6}\right)^r - \ldots\right\}.$$

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But $\int_0^{\pi} \cos^{2n+1}\phi\, d\phi = 0$, and $\int_0^{\pi} \cos^{2n}\phi\, d\phi = 2\int_0^{\frac{1}{2}\pi} \cos^{2n}\phi\, d\phi$

$$= \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \cdot \pi = \frac{\Pi(2n)}{2^{2n}\{\Pi(n)\}^2} \cdot \pi;$$

thus the integral becomes

$$\frac{\sqrt{3}}{\sqrt{2}} \int_0^{\frac{1}{2}\pi} d\phi \left\{1 + \frac{11}{\Pi(2)}\left(\frac{\cos\phi}{6}\right)^2 + \frac{3.19}{\Pi(4)}\left(\frac{\cos\phi}{6}\right)^4 + \frac{3.5.7.27}{\Pi(6)}\left(\frac{\cos\phi}{6}\right)^6 \right.$$

$$\left. + \frac{3.5.7.9.11.35}{\Pi(8)}\left(\frac{\cos\phi}{6}\right)^8 + \ldots + \frac{3.5.7\ldots(4n-5).(8n+3)}{\Pi(2n)}\left(\frac{\cos\phi}{6}\right)^{2n} + \ldots\right\}$$

$$= \frac{\pi\sqrt{3}}{2\sqrt{2}}\left\{1 + \frac{11}{12^2} + \frac{3.19}{\{\Pi(2)\}^2 12^4} + \frac{3.5.7.27}{\{\Pi(3)\}^2 12^6} + \frac{3.5.7.9.11.35}{\{\Pi(4)\}^2 12^8} + \ldots \right.$$

$$\left. \ldots + \frac{3.5.7\ldots(4n-5).(8n+3)}{\{\Pi(n)\}^2 12^{2n}} + \ldots\right\}.$$

Putting t_r for the rth term of this series, we have

$$\frac{t_{r+2}}{t_{r+1}} = \frac{1}{9} \cdot \frac{(4r-3)(4r-1)(8r+11)}{(4r+1)(4r+1)(8r+3)};$$

whence we see that the series is convergent, and that the ratio of the $(r+2)$th term to the $(r+1)$th approaches the limit $\frac{1}{9}$ as r increases; thus, after summing a certain number (r say) of the terms, we can obtain a near approximation to the sum of all the remaining terms by treating the rest of the series as a geometrical progression whose common ratio is $\frac{1}{9}$; that is to say, $t_r + t_{r+1} + t_{r+2} + \ldots$ ad inf. $= \frac{t_r}{1-\frac{1}{9}} = \frac{9}{8}t_r = t_r + \frac{1}{8}t_r$.

The value of the integral is, therefore,

$$\Sigma = -\tfrac{1}{2} + \frac{3\pi\sqrt{6}}{16}\left\{1 + \frac{11}{2^4.3^2} + \frac{19}{2^{10}.2^3} + \frac{35}{2^{14}.3^4} + \frac{13475}{2^{22}.3^7} + \ldots\right\}$$

$$= -1{\cdot}5 + \tfrac{3}{16}\pi\sqrt{6}\left\{1{\cdot}0763889 + {\cdot}0007872 + {\cdot}0000264 \right.$$

$$\left. + {\cdot}0000015 + (\tfrac{1}{8} \text{ of } {\cdot}0000015 = {\cdot}0000002)\right\}$$

$$= -1{\cdot}5 + \tfrac{3}{16}(1{\cdot}077104)(2{\cdot}449489)(3{\cdot}14159)$$

$$= 1{\cdot}55412 - 1{\cdot}5 = {\cdot}05412,$$

which agrees to the last decimal place with the value previously found.

From this the value of the mean area of the complete ellipses is found to be ·26409, which is a little less than one-sixth of the area ($\pi \sin^2 \frac{1}{4}\pi = \frac{1}{2}\pi = 1{\cdot}5707$) of the base.

4190. (Proposed by C. Leudesdorf.)—A circular target is divided into three rings by drawing (1) a concentric circle within it, (2) a second concentric circle within the last; the radii of (1) and (2) being taken at random. If two arrows be shot so as both to strike the target, show that the chances that both lie within the same ring, that they lie in adjacent rings, and that they lie in non-adjacent rings are $\frac{1}{12}\frac{2}{5}\frac{5}{6}$, $\frac{4}{15}\frac{3}{0}$, $\frac{1}{6}\frac{3}{0}$.

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Solution by S. FORDE.

If α, β, γ be the areas of the rings, the chances are

$$\frac{p_1}{\alpha^2+\beta^2+\gamma^2} = \frac{p_2}{2\beta(\alpha+\gamma)} = \frac{p_3}{2\alpha\gamma} = \frac{1}{(\alpha+\beta+\gamma)^2}.$$

And if r, r', c, be the radii of the three concentric circles

$$\alpha = \pi r^2, \quad \beta = \pi(r'^2-r^2), \quad \gamma = \pi(c^2-r'^2);$$

therefore $p_1 = \int_0^c \frac{dr}{c} \int_r^c \frac{r^4+(r'^2-r^2)^2+(c^2-r'^2)^2}{c^4} \frac{dr'}{c-r} = \frac{128}{225}$,

$$p_2 = \int_0^c \frac{dr}{c} \int_r^c \frac{2}{c^4}(r'^2-r^2)(r^2-r'^2+c^2) \frac{dr'}{c-r} = \frac{43}{150},$$

$$p_3 = \int_0^c \frac{dr}{c} \int_r^c \frac{2}{c^4} r^2(c^2-r'^2) \frac{dr'}{c-r} = \frac{13}{90}.$$

4140. (Proposed by Professor TOWNSEND, F.R.S.)—A quartic curve being supposed determined on a ruled quadric by the intersection with it of any other quadric surface; prove that generally,

(*a*.) The problem, to construct, with generators of the quadric, a skew polygon of any even order inscribed to the quartic, is either indeterminate or impossible.

(*b*.) Of the several generators of the quadric, two tetrads of opposite systems touch the quartic, and are equi-anharmonic with each other.

Regarding a binodal plane quartic curve as the projection of a twisted quartic in space; show immediately from the preceding (see SALMON'S *Higher Plane Curves*, 2nd ed., arts. 282 and 270) that,

(*a'*.) The problem, to construct, with nodal chords of the curve, a polygon of any even order inscribed to it, is either indeterminate or impossible.

(*b'*.) Of the several nodal lines of the curve, two tetrads through opposite nodes are tangents to it, and are equi-anharmonic with each other.

Solution by the PROPOSER.

To prove the first property. Denoting by U the ruled and by V the intersecting quadric, by W any one of the four cones passing through UV, and by W' the cone which from the same vertex envelopes U; then, every generator of U of either system being a tangent to W', since consequently the several planes connecting with A the several sides of any polygon of generators inscribed to UV determines a polyhedral angle at once inscribed to W and circumscribed to W', therefore &c., as regards (*a*); and, every generator of U of either system which touches UV being a tangent at once to W and to W', and the four planes tangent to both cones determining two tetrads of generators of opposite systems intersecting in corresponding pairs

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at four complanar points on the curve of contact of W′ with U, therefore, &c., as regards (*b*).

To prove the second property. Denoting by U the quadric of the system passing through the quartic in space which passes through the centre of projection O, by S the projected plane quartic, by E and F the two generators of U passing through O, and by M and N their two intersections with the plane of projection, that is, the two nodes of S; then, since every chord of S passing through M or N is the projection of a generator of U intersecting E or F, therefore, &c., from the preceding, as regards both (*a*′) and (*b*′).

Cor. 1.—As four cones W, X, Y, Z pass through the intersection UV of any two quadric surfaces U and V, and as to each of them corresponds a cone having the same vertex enveloping either of them U, there are consequently [as is otherwise evident on the fundamental principles of anharmonic section] four different modes of correspondence of constituents for which the two tetrads of tangent generators in the general property (*b*), and of tangents through the nodal points in the general property (*b*′), are equi-anharmonic with each other.

Cor. 2.—The four vertices A, B, C, D of the four cones passing through the intersection UV of any two quadric surfaces U and V determining the tetrahedron self-reciprocal to both, it follows also from the general property (*b*) above [as may be readily proved directly *à priori*] that when two tetrads of generators of opposite systems of any ruled quadric are equi-anharmonic, their sixteen intersections lie by fours in four planes, which determine a self-reciprocal tetrahedron with respect to the surface.

Cor. 3.—The binodal plane quartic S, in the second general property, degenerating into a non-singular plane cubic when the centre of projection O is on the twisted quartic UV, and the two generators E and F of the concurrent quadric through O becoming in the same case any two chords of UV conterminous with O and complanar with the tangent to UV at O, that property is consequently true also, in both its parts, for a non-singular plane cubic, with respect to any two points of it, M and N collinear with that on it determined by the tangent to the projecting quartic at the centre of projection.

4235. (Proposed by the Rev. W. Roberts, M.A.)—Find the locus of points on the ellipses of a confocal system, at which the tangents cut off a constant area from the axes. A system (A) of curves being obtained by supposing the constant area to take different values, it is required to determine the system which cuts (A) orthogonally.

Solution by Colonel Clarke, C.B., F.R.S.

Let the equation of the system of ellipses be $\frac{x^2}{a^2+1} + \frac{y^2}{a^2} = 1$; then the double area cut off from the axes by a tangent at the point xy is $n = \frac{a^2(a^2+1)}{xy}$; consequently $a^4 + a^2 = nxy = a^2(x^2+y^2) + y^2$.

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The elimination of e^2 from these equations gives

$$(x^2+y^2)^2-x^2+y^2-\left(n-\frac{1}{n}\right)xy=0,$$

or
$$r^2-\cos 2\theta-\frac{1}{2}\left(n-\frac{1}{n}\right)\sin 2\theta=0 \quad \ldots\ldots\ldots\ldots (1),$$

which is the required locus, evidently a lemniscate concentric with the ellipses and passing through their common foci; the axis of the lemniscate making with the axis of x an angle α such that $2\tan 2\alpha = n-\frac{1}{n}$.

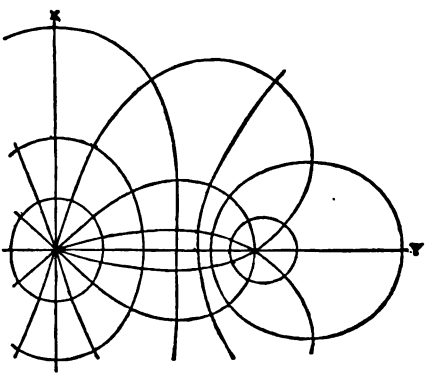
The equation (1) gives

$$\frac{dr}{r\,d\theta}-\frac{1}{r^3}(-\sin 2\theta+\tan 2\alpha\cos 2\theta)=0 \quad \ldots\ldots\ldots\ldots (2).$$

Substituting here $-\frac{r\,d\theta}{dr}$ for $\frac{dr}{r\,d\theta}$, and eliminating $\tan 2\alpha$ between (1) and (2), the result is
$$\frac{r\,d\theta}{dr}\sin 2\theta+\cos 2\theta-\frac{1}{r^2}=0,$$

which gives $\frac{1}{c}-2\frac{\cos 2\theta}{r^2}+\frac{1}{r^4}=0$, or $r^4-2cr^2\cos 2\theta+c=0$ ..... (3),

as the orthogonal trajectory of the curves (1).



As c increases negatively from 0 to $-\infty$, the closed curve (3), which is initially a small circle round the origin, expands; and the value of r for $\theta=0$ increases from 0 to $\pm\frac{1}{\sqrt{2}}$, and that for $\theta=\frac{1}{4}\pi$ from 0 to $\pm\infty$. When c is a very large negative quantity, this loop assimilates to a very large lemniscate (axis, $\theta=\frac{1}{4}\pi$), though without crossing at the origin. For $c=\pm\infty$ the curve becomes an equilateral hyperbola confocal with the ellipses; and as c proceeds from ∞ to 1, the two similar loops of which (3) now consist diminish to indefinitely small circles round the foci. The curves shown in the figure correspond to the following values:—

$$c=-\cdot00375,\ -\cdot06116,\ -\cdot4639,\ \infty,\ 1\cdot800,\ 1\cdot115.$$

4270. (Proposed by Professor CROFTON, F.R.S.)—1. The resultant of a system of forces acting in the same sense along the sides of a portion AF of a polygon, and whose magnitudes are respectively m.AB, n.BC, p.CD, q.DE, r.EF, is, in magnitude and direction (not position), r.GF; where G

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is the centre of gravity of a system of weights represented by m, $n-m$, $p-n$, $q-p$, $r-q$; placed respectively at A, B, C, D, E. Some of these may of course be *negative* weights. Likewise for any number of forces.

2. Tangential stress acts along any arc AB of a curve, of intensity always proportional to the distance from A measured along the curve. Shew that the resultant is in magnitude and direction, i. GB; where i is the intensity of the stress at B, and G is the centre of gravity of the arc AB.

I. *Solution by* Professor TAIT.

1. Let $\alpha, \beta, \ldots \zeta$ be the sides in order. Then the resultant is

$$m\alpha + n\beta + p\gamma + \ldots + q\epsilon + r\zeta,$$

and this may obviously be written as

$$r\left((\alpha+\beta+\ldots+\zeta) - \frac{(n-m)\,\alpha + (p-n)\,(\alpha+\beta) + \ldots + (r-q)\,(\alpha+\beta+\ldots+\epsilon)}{r}\right)$$

which is the theorem.

2. Let s be the arc to any point P from A, ρ the vector of P. Then the resultant is

$$h\int_0^{s_1} s\,d\rho = hs_1\rho_1 - h\int_0^{s_1}\rho\,ds = hs_1(\rho_1 - \bar{\rho}),$$

where $\bar{\rho}$ is the vector of the centre of inertia of the arc s_1. This, of course, like the former theorem, does not require restriction to a plane.

II. *Solution by* J. J. WALKER, M.A.

1. Take, on AB, $A'B = m.AB$; and, on BC, $BC' = n.BC$, then $A'C'$ represents in magnitude and direction the resultant of the forces acting along AB, BC respectively. Draw CB_1 parallel to $C'A'$ and meeting AB in B_1; then $BB_1 : BA' = 1 : n$, therefore $BB_1 : BA = m : n$, therefore B_1 is the centre of gravity of the weights supposed to act at A and B. Also $B_1C : A'C' = 1 : n$, therefore $n.B_1C = A'C'$. Hence, instead of the forces acting along AB, BC, it may be supposed that a force acts along B_1C equal to nB_1C; and instead of the weights at A, B, that there is a weight at B_1 represented by $m+n-m = n$. Combining similarly the forces along B_1C and CD it follows that their resultant is represented in magnitude and direction by $p.C_1D$, where C_1 is that point on B_1C which is the centre of gravity of the weights at A, B, C; and so on for the successive sides of the polygon, how many soever.

2. Suppose the sides to become all equal, and the forces along AB, BC, CD, ... EF, to be represented by $i.AB$, $2i.AB$, $3i.AB$, ...; then the weights at A, B, ... F will all be equal, and G will be the centre of gravity of the part of the perimeter ABC ... F. In the limit, when AB, BC, ... become infinitesimal, the result as enunciated follows, reading F for B.

In the general case it will be observed that unless $m, n, p, \ldots r$ were all positive and formed an ascending series in magnitude, one or more of the points $B_1, C_1, \ldots$ G would only be centres of gravity in the extended sense defined in the question.

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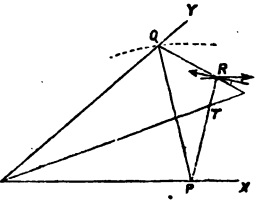
4172. (Proposed by Professor WOLSTENHOLME.)—If two points fixed in a lamina slide upon two straight lines fixed in space (in one plane), and if one point be made to oscillate backwards and forwards so as to have always the same velocity, the ellipse described by any fixed point of the lamina will be described under acceleration which is fixed in direction.

Solution by R. W. GENESE, B.A.



Let the points P and Q move along OX and OY; P with uniform velocity u; and let R be the point in the lamina considered. The velocity of R is made up of u parallel to OX, and PR $\frac{d\text{RPO}}{dt}$ perpendicular to PR.

Again, the velocity of Q is made up of u parallel to OX and PQ $\frac{d\text{QPO}}{dt}$ perpendicular to PQ; but Q moves along QO, therefore PQ $\frac{d\text{QPO}}{dt}$ cos OQP = u sin POQ, therefore $\frac{d\text{QPO}}{dt}$ cos OQP is constant.

Make the angle QOT = RPQ, therefore the angle OTP = OQP.

If the velocity of R perpendicular to PR be resolved into two parallel and perpendicular to OT, the latter is seen to be constant; and the velocity of R parallel to OX can be resolved into two constant velocities parallel to these directions. Therefore the actual velocity of R perpendicular to OT is constant; therefore R moves with an acceleration parallel to OT.

4201. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—Prove (1) that the four points of contact of a twisted quadrilateral with a surface of the second order lie in a plane; and (2) that the tangent planes containing the sides of the quadrilateral meet in one point, viz., the pole of this plane.

I. *Solution by* R. W. GENESE, B.A.

1. Let $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ be the equations to the planes containing pairs of adjacent sides of quadrilateral. If we write down the general equation to a quadric referred to these planes, and then make it touch the lines $\left.\begin{matrix}\alpha=0\\ \gamma=0\end{matrix}\right\}$, $\left.\begin{matrix}\alpha=0\\ \delta=0\end{matrix}\right\}$, $\left.\begin{matrix}\beta=0\\ \gamma=0\end{matrix}\right\}$, $\left.\begin{matrix}\gamma=0\\ \delta=0\end{matrix}\right\}$, we find that it will reduce to the form

$$(p\alpha+q\beta+r\gamma+s\delta)^2 = m\alpha\beta+n\gamma\delta.$$

The above lines meet this quadric in points on the plane

$$p\alpha+q\beta+r\gamma+s\delta = 0.$$

2. The second part follows at once from the first.

[The theorem in (1) is a particular case of that given by Mr. MERRIFIELD in Question 2642, and proved by him on p. 77 of Vol. XI. of the *Reprint*.]

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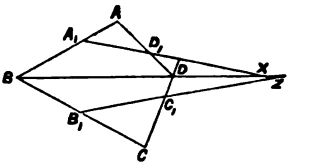
II. *Solution by the* PROPOSER.



Since ABCD is a quadrilateral circumscribing a surface of the second order we must have by Carnot's theorem the following relation between the segments of the tangents

$$AA_1 . BB_1 . CC_1 . DD_1 = AD_1 . BA_1 . CB_1 . DC_1 \quad \ldots\ldots (a).$$

In the triangle BAD draw the transversal A_1D_1X, then we shall have

$$AA_1 . DD_1 . BX = AD_1 . BA_1 . DX \quad \ldots\ldots (b).$$

Dividing (a) by (b) $$\frac{BB_1 . CC_1}{BX} = \frac{CB_1 . DC_1}{DX} \quad \ldots\ldots (c).$$

In like manner, if we draw a transversal through the points B_1, C_1 in the triangle BCD meeting the diagonal BD in Z, we shall have

$$BB_1 . CC_1 . DZ = CB_1 . DC_1 . BZ \quad \ldots\ldots (d).$$

Dividing (d) by (c), we shall have $DZ . BX = BZ . DX$, or $BX : DX = BZ : DZ$, or $BD : DX = BD : DZ$, or X coincides with Z. Consequently a plane may be drawn through the points $XA_1B_1C_1D_1$.

And since the points $A_1B_1C_1D_1$ of the twisted quadrilateral are in a plane, this plane may be taken as the base of the cone circumscribing the surface of the second order, and accordingly the tangent planes to the surface through the sides of the quadrilateral will be tangent planes to the cone, and therefore meet in its vertex.

4265. (Proposed by Sir JAMES COCKLE, F.R.S.)—Boole (*Differential Equations*, 2nd ed., p. 350) says that the method of Charpit supposes that q can be expressed as a function of x, y, z, p. He subsequently (p. 351) notes that it presupposes the existence of two algebraic equations between x, y, z, p, q. Explain this.

Solution by the PROPOSER.

The supposition that the equations are algebraic seems to be unnecessary. Thus, taking an example useful in other respects, let $q = e^{-\frac{p}{A}}$, then (ib., p. 337) the auxiliary system is

$$Ae^{\frac{p}{A}} dx = dy = \frac{Ae^{\frac{p}{A}} dz}{p + A} = \frac{dp}{0};$$

whence $$x = \frac{z}{p + A} + c, \text{ and } (p + A)(x - c) = z.$$

Hence, since $p dx + q dy = dz$, we pass to

$$\left(\frac{z}{x - c} - A\right) dx + e^{1 - \frac{z}{A(x-c)}} dy = dz,$$

which is equivalent to

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$$\frac{edy}{A} = \frac{1}{A} e^{\frac{z}{A(x-c)}} \left\{ dz - \frac{zdx}{x-c} + Adx \right\} = d\left\{ (x-c) e^{\frac{z}{A(x-c)}} \right\}.$$

Therefore, integrating, $\frac{ey}{A} + b = (x-c) e^{\frac{z}{A(x-c)}}$,

and $\frac{z}{A(x-c)} = \log \frac{ey+Ab}{A(x-c)}$, or $z = A(x-c) \log \frac{ey+Ab}{A(x-c)}$,

a complete solution containing the two arbitrary constants b and c. This solution gives $p = \frac{z}{x-c} - A$ and

$$q = A(x-c) \frac{e}{ey+Ab} = e^{1-\frac{z}{A(x-c)}} = e^{-\frac{p}{A}},$$

and is thus verified.

4263. (Proposed by G. M. Minchin, M.A.)—A rigid body is acted on by any system of forces. If from any point on Poinsot's Central Axis a right line be drawn perpendicular to this axis, show that the principal axes of moments at the various points along this right line trace out a hyperbolic paraboloid.

Solution by E. B. Elliott.

Let R be the resultant force along, and G the resultant moment round the central axis. Take this as axis of z, and the chosen line perpendicular to it as that of x.

At any distance a along this line the component moments are G round a parallel to z-axis, and Ra round a parallel to y-axis; therefore the axis of principal moment is the intersection of the planes

$$x = a \text{ and } \frac{z}{G} = \frac{y}{Ra};$$

therefore eliminating a it lies on the surface $Gy = Rzx$, a hyperbolic paraboloid.

4298. (Proposed by J. W. L. Glaisher, B.A.)—With four given straight lines to form a quadrilateral inscribable in a circle.

I. *Solution by* Professor Cayley.

Let the sides of the quadrilateral taken in order be a, b, c, d; and let its diagonals be x, y; viz., x the diagonal joining the intersection of the sides

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a, b with that of the sides c, d; y the diagonal joining the intersection of the sides a, d with that of the sides b, c; then, the quadrilateral being inscribed in a circle, the opposite angles are supplementary to each other. Suppose for a moment that the angles subtended by the diagonal x are $\theta, \pi - \theta$, we have

$$x^2 = b^2 + c^2 + 2bc\cos\theta, \quad x^2 = a^2 + d^2 - 2ad\cos\theta;$$

and thence $(ad+bc)\,x^2 = ad\,(b^2+c^2) + bc\,(a^2+d^2) = (ac+bd)(ab+cd)$,

that is, $x^2 = (ac+bd)\,\dfrac{ab+cd}{ad+bc}$, and similarly $y^2 = (ac+bd)\,\dfrac{ad+bc}{ab+cd}$,

agreeing as they should do with the known relation $xy = ac+bd$: the quadrilateral is thus determined by means of either of its diagonals. It is however interesting to treat the question in a different manner.

Considering a, b, c, d, x, y as the sides and diagonals of a quadrilateral, we have between these quantities a given relation, say

$$F(a, b, c, d, x, y) = 0,$$

and the quadrilateral being inscribed in a circle, we have also the relation $xy = ac+bd$; which two equations determine x, y; and thus give the solution of the problem.

The expression of the function F is in effect given in my paper, "Note on the value of certain determinants, &c.," *Quarterly Mathematical Journal*, t. iii. (1859), pp. 275—277; viz., a, b, c being the edges of any face, and f, g, h the remaining edges of a tetrahedron, then

$$\begin{aligned}\text{volume} = \tfrac{1}{144}\big\{ & b^2c^2(g^2+h^2) + c^2a^2(h^2+f^2) + a^2b^2(f^2+g^2) \\ & + g^2h^2(b^2+c^2) + h^2f^2(c^2+a^2) + f^2g^2(a^2+b^2) \\ & - a^2f^2(a^2+f^2) - b^2g^2(b^2+g^2) - c^2h^2(c^2+h^2) \\ & - a^2g^2h^2 - b^2h^2f^2 - c^2f^2g^2 - a^2b^2c^2\big\},\end{aligned}$$

where, when the tetrahedron becomes a quadrilateral, the volume is $=0$.

In this formula, changing c, b, h, g, f, a into a, b, c, d, x, y, we have the required equation $F=0$; viz., this is found to be

$$\begin{aligned}& a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 - b^2d^2(b^2+d^2) - a^2c^2(a^2+c^2) \\ & \qquad + x^2y^2(a^2+b^2+c^2+d^2-x^2-y^2) \\ & + x^2(a^2c^2+b^2d^2-a^2d^2-b^2c^2) + y^2(a^2c^2+b^2d^2-a^2b^2-c^2d^2) = 0,\end{aligned}$$

which, with $xy = ac+bd$, determines x, y. Substituting in the foregoing equation for xy its value, the equation becomes

$$\begin{aligned}(ad+bc)^2x^2 + (ab+cd)^2y^2 = 2\big\{ & a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 \\ & + abcd(a^2+b^2+c^2+d^2)\big\},\end{aligned}$$

or $$(ad+bc)^2x^2 + (ab+cd)^2y^2 = 2(ad+bc)(ab+cd)(ac+bd).$$

To show more clearly how this equation arises, I observe that we have identically

$$\begin{aligned}F - (a^2+b^2+c^2+d^2-x^2-y^2)(xy+ac+bd)(xy-ac-bd) \\ = 2(ad+bc)(ab+cd)(xy-ac-bd) \\ = \big\{(ad+bc)x - (ab+cd)y\big\}^2.\end{aligned}$$

The resulting equation $(ad+bc)x - (ab+cd)y = 0$, together with $xy = ac+bd$, gives for x, y the foregoing values.

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II. *Solution by* H. MURPHY; CHRISTINE LADD; *and others.*

By Ptolemy's theorem the rectangle of the diagonals is given; and their ratio is, by a well-known property, the same as the ratio of the sums of the rectangles contained by the sides terminating on them;* hence both the ratio and the rectangle of the diagonals become known, and the required quadrilateral is determined by inflecting the given sides from the ends of either diagonal.

NOTE ON QUESTION 4298. *By* J. W. L. GLAISHER, B.A.

I was anxious that this Question should appear in the *Educational Times*, not only for its intrinsic interest, but also on account of its historic celebrity. There are not many problems which have excited greater interest than did this at the end of the sixteenth and beginning of the seventeenth centuries. In Van Ceulen's *De Circulo et Adscriptis*, which was edited by Snell, no less than ten pages are devoted to its discussion, and it occupied the attention of Vieta, Adrianus Romanus, Van Ceulen, Snell, John Prætorius, and Scaliger. There are in the *De Circulo* three solutions, of which I propose to give a brief account, as also of the incidental matters and points of history involved. In order to render intelligible what follows it is necessary to bear in mind that Van Ceulen died in 1610, and that in 1615 was published his posthumous work *De Arithmetische en Geometrische Fondamenten* (fol. Leyden), written in Dutch. A Latin translation by Snell, entitled *Fundamenta Arithmetica et Geometrica* (4to, Leyden) appeared in the same year, and in 1619 a second edition was published under the title of *De Circulo et Adscriptis Liber* (4to, Leyden). The last two works are essentially the same, the contents being a treatise on surds, geometrical problems, and the tract *De Circulo*, which in the 1619 edition gives its name to the whole work. There were other editions or reprints that need not be referred to here. In 1596 Van Ceulen had published his *Van den Circkel* (fol. Delf), with which we are not here concerned, as it contains no allusion to the problem of the four lines in a circle. We thus have the Dutch folio of 1615, containing, no doubt, what Van Ceulen wrote, as he left it; and the Latin quarto of the same date, containing a translation altered and amended, and with the addition of Snell's own (copious) notes. I shall follow the 1615 (or, what comes to the same thing, the 1619) Latin quarto, and the Dutch edition will only be referred to incidentally.

It is convenient here to write down the ordinary known formulæ for the angles and diagonals of a quadrilateral inscribed in a circle; viz., if a, b, c, d be the sides DA, AB, BC, CA of the quadrilateral, then

$$\sin A = \frac{S}{4(ab+cd)}, \quad \sin B = \frac{S}{4(ad+bc)},$$

where $S^2 = (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)$,

* This Theorem is given as Proposition F, in Thomson's Euclid, ed. 1864, p. 157; and in a note to this proposition a mode of construction and calculation of Question 4298 is pointed out.

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and $BD^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}$, $AC^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}$.

In what follows, the given straight lines will, for brevity, be taken to be a, b, c, d; and the same letters (viz. DA = a &c.) will be assumed as above. After giving the constructions of the different solutions, it will be sufficient to show that they lead to the above formulæ, and so dispense with the prolix geometrical proofs. It is scarcely necessary to observe that the construction of the problem is effected when one of the diagonals is found, or one of the angles.

Van Ceulen begins the fourth book of his "Problemata Miscellanea," (the headings of the pages running, however, in the 1615 edition, "Ludolphi à Ceulen Lib. V. De Problematis Miscellanis (*sic*)," and the "Lib. V." being omitted in the 1619 edition) with the problem in question, viz., *E datis quatuor lineis quadrangulum construere quod sit in circulo.* He states that, twenty years before, John Pauli had proposed the problem to him, when he solved it by Algebra, and afterwards investigated a method which those ignorant of Algebra might follow. The rule (which he only applies to the case when the lines are 18, 9, 8, 6) stated generally is:—Form $\frac{b}{d} + \frac{a}{c}$, and $\frac{ba}{cd} + 1$, and multiply them together [thus obtaining $\frac{(ad+bc)(ab+cd)}{c^2d^2}$], form $ac + bd$, divide it by the product already found and extract the square root [viz. $\frac{cd\sqrt{(ac+bd)}}{\sqrt{(ad+bc)}\sqrt{(ab+cd)}}$]. Multiply this quantity by that which was formed second, and the result is the diagonal, [viz.,

$$\frac{cd\sqrt{(ac+bd)}}{\sqrt{(ad+bc)}\sqrt{(ab+cd)}} \cdot \frac{ab+cd}{cd} = \frac{\sqrt{(ac+bd)}\sqrt{(ab+cd)}}{\sqrt{(ad+bc)}} = AC,$$

which is the case.] This rule Van Ceulen says is his own (inventum meum), and he adds that the "ingenious youth" Cornelius Petreius of Alcmaar four years before solved the question geometrically. He then proceeds to give this solution, which is as follows:—Find a mean proportional between a and b, and also one between c and d; take these mean proportionals as sides of a rectangle, and find the square on the diagonal $[ab+cd]$. Treat similarly a, d and b, c $[ad+bc]$; and a, c and b, d $[ac+bd]$. Then find a fourth proportional to these three squares $\left[\frac{(ac+bd)(ad+bc)}{ab+cd}\right]$, and it will be the square on the diagonal BD. No proof whatever is given of either Van Ceulen's or Petreius's method.

Snell here adds a long note of his own. He says that the problem was made famous (nobilitatum) by the bad mathematics (pseudographemate) of Joseph Scaliger, who endeavoured to persuade the world that he had squared the circle, and thereby invited others to the solution of the question; whereupon Vieta successfully accomplished it, and Joachim Prætorius, unsurpassed in such matters, published a whole book about the four lines in the circle, in which he cleverly showed in many ways that the problem could be solved (clarissimus Ioachimus Prætorius, harum artium scientia nulli secundus, de quatuor lineis in circulo integrum librum publicavit, in quo multis modis ingeniose sane et acute hoc idem

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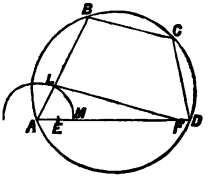
problema effici posse demonstravit). He then says that long ago he himself, incited by Van Ceulen, obtained an analysis of the question; and as it throws light on the rules that have been given, he appends it without demonstration. The analysis consists in showing that the ratio of the diagonals is as $ad+bc$ to $ab+cd$, while Ptolemy had demonstrated [in the theorem which Simson added to the sixth book of Euclid as proposition D] that the product of the diagonals was equal to $ac+bd$. These theorems of course lead to the values of BD and AC given above, and justify Van Ceulen's and Petreius's rules. Snell remarks that the synthesis of the problem is easy, and after some additional observations he proceeds to a second solution which he says is "satis opportuna," because some years before, when he was returning from Prague with Adrianus Romanus and was visiting John Prætorius at Altorf, he was invited to the discussion of the matter by his friends, and especially Prætorius. They obtained (excogitaverunt) three solutions, of which the simplest was that which he then gives, and which may be stated thus:—Let b be greater than d; take GH a fourth proportional to $b-d$, d, $a+c$, and divide GH in E so that GE is a fourth proportional to $b+d$, GH$+a$, d. Form EG, EH, d, into a triangle E′G′H′, and produce E′G′ to M making E′M $= c$, and E′H′ to N making E′N $= a$; then G′MNH′ is the quadrilateral required. The reason for this construction is best seen as follows:—

$$\text{GH} = \frac{(a+c)\,d}{b-d}, \quad \therefore \text{EG} = \frac{d}{b+d}(\text{GH}+a) = \frac{d(ab+cd)}{b^2-d^2}, \text{ and EH} = \frac{d(ad+bc)}{b^2-d^2}.$$

We have therefore to show that two of the angles of the triangle whose sides are $ad+bc$, $ab+cd$, b^2-d^2 are equal to two angles of the required quadrilateral. To demonstrate this, call the triangle A′B′C′, and we observe that $2s' = (b+d)(a+b+c-d)$, $2s'-2a' = (b-d)(a+b-c+d)$,

$$2s'-2b' = (b-d)(-a+b+c+d), \quad 2s'-2c' = (b+d)(a-b+c+d);$$

whence it is easily seen that sin A′ and sin B′ have the same values as those written down above for sin A and sin B in the quadrilateral. This second solution concludes Snell's note.



In his third paragraph Van Ceulen proceeds to give Vieta's solution of the question. He says that, many years before, when he learnt from Euclid that the opposite angles of an inscribed quadrilateral were equal to two right angles, he had fretted himself a great deal about the solution of the question, but met with no success. Vieta however had solved it in his work of 1595, and he reproduces Vieta's proposition of which the construction is as follows:—let a be greater than c, and b greater than d; take DA $= a$, and divide $a-c$ in the ratio of b to d; let AE be the greater and DF the less segment. With centre A and radius $b-d$ draw the semicircle ML, and from F draw FL to the semicircle equal to ED. Produce AL to B making AB $= b$, draw BC parallel to LF and join CD; then CD shall be equal to d, and ABCD shall be in a circle. It is easily seen algebraically that this construction is correct, and that the angles A and L of the triangle ALF are equal to those at A and B of the required quadrilateral; for

$$\text{AE} = \frac{b}{b+d}(a-c), \text{ DF} = \frac{d}{b+d}(a-c), \text{ AF} = \text{AE}+c = \frac{ab+cd}{b+d}, \text{ DE} = \frac{ad+bc}{b+d},$$

and AL $= b-d$, so that the sides of the triangle ALF are proportional to $ab+cd$, $ad+bc$, b^2-d^2, viz., to the sides of the triangle A′B′C′, just used in

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the verification of Snell's second construction. Vieta's construction and his proof are both purely geometrical (and they have not the appearance of "Algebra translated" which characterises those of Snell), and his treatment of the question is worthy of his reputation. The proposition will be found in the *Pseudo-Mesolabum et alia quædam adjuncta Capitula*, on pp. 277—279 of the Elzevir edition of his collected works (1646). He has there discussed the matter very fully, pointing out the different cases when certain of the sides are equal, &c. I should mention that in this *résumé* I have merely endeavoured to give an idea of the different constructions, and have omitted details as to which side should be opposite to which, &c.

It will have been noticed that Van Ceulen's statements do not appear perfectly consistent. He first says Pauli proposed the problem to him, and subsequently his words imply that it occurred to him on reading Euclid. But this is of no importance: it is clear that he solved the question algebraically, and so obtained a rule; but that he was unable to translate the algebraical work into geometry, and prove the rule. The exact words in his third paragraph are worth quoting; they are: "... hic me multum excruciassem modo hanc modo istam viam tentando, ut aliquo saltem modo propositum problema solverem, fateor me irrito conatu omnia expertum, neque quidquam profecisse. Idem tamen toties à me frustra tentatum .... Vieta .... solvit." In the Dutch edition, his words are, "... dit bedencken voor veel jaren heest mijn grooten arbeyt ghecost om door d'een ende ander middelen *Geometrici* te vinden het voorgaende, maer niet uytghericht, welck ick bekenne. Dese vvetenschap ick vergheefs ghesocht .... is ghevonden .... door .... Vieta." It is the omission of the word *Geometrici* (emphasised by Van Ceulen) in Snell's translation that there renders the meaning somewhat ambiguous, until it is taken in connexion with the context. "Excruciassem" seems a rather strong rendering of "grooten arbeyt ghecost," but it must be remembered that Snell was intimately acquainted with Van Ceulen, and he probably used the word "excrucio" advisedly. Van Ceulen could not write Latin, and this placed him at a disadvantage in his relations with other mathematicians. John Pauli appears in Dutch as *Meester Iohan Pouwelsz*, and is described as "ein recht Liebhebber der const, ende vvel hervaren in dien," (a true lover of the art, and well skilled in it). Snell describes him as "non ignobilis Geodæta" the last word meaning, no doubt, a Geometrician, and not, as at first sight might seem likely, a surveyor. Cornelius Petreius of Alcmaar (in Dutch *Cornelisz Pietersz*) was a pupil of Van Ceulen's who helped him in his calculation of π. A certain *NiclaesPietersz* of Deventer is mentioned as a mathematician in the *Van den Circkel*, who perhaps was his father. It is well known that Joseph Scaliger, when young, published a false quadrature of the circle, in a book entitled "Cyclometrica Elementa," Leyden, 1594, and was refuted by Vieta; I have not seen the work, and do not know how the four-line problem is alluded to in it. The John Prætorius who, Snell states, wrote a whole book on the subject, is no doubt the astronomer of that name (born at Joachimstal 1537, died at Altorf 1616). I have made no special search for the book in question, but it is not mentioned by Heilbronner, Murhard, Dechales, nor (I think) by Kastner. Snell speaks of him the first time as *Joachim* Prætorius, no doubt by a slip of the pen; but although none of the biographical dictionaries contain such a name, there was a Joachim Prætorius living at this time who published at Hamburg, in 1607, a work entitled *Historia Elephanti*, a copy of which is in the British Museum. There can be little doubt the Altorf referred to, where Snell, Romanus, and Prætorius discussed the problem, was the Bavarian

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Altorf near Nüremberg, where there used to be a University; but there are two other Altorfs that are *possible*, viz., one in Würtemberg and another in Baden.

There are not many isolated problems which bring into close relation five more remarkable contemporary names than those of Vieta, Snell, Romanus, Van Ceulen, and Scaliger.

In the Dutch edition of 1615, the problem occupies pp. 203—208. In the Latin edition of the same date it occupies pp. 185—194, and in the Latin edition of 1619 it follows p. 122, and occupies the pages numbered 185, 136, 137, 138, 132, 136, 141, 122, 143. 144 (these being ten consecutive pages). The typography of the Dutch work is excellent, and that of the Latin work of 1619 (as the specimen of pagination just quoted indicates) abominable. In both the Latin editions the first page is from the same type, but the rest was reset for the edition of 1619; some errors are corrected, and some additional ones introduced. It is proper to state that Snell heads Van Ceulen's paragraphs (or articles, as we should now call them) *Problema* 1, *Problema* 2, &c., thus giving the investigation an absurd appearance, as Problems 1, 2 and 3 are all the same.

4234. (Proposed by the Editor.)—1. Given the base of a triangle, and (α) the difference of the angles at the base, or (β) that the bisector of the vertical angle is parallel to a fixed straight line, or (γ) that the sides make equal angles with two fixed straight lines; prove that the locus of the vertex will be a rectangular hyperbola, having its centre at the middle point of the base, and its asymptotes parallel to the bisectors of the vertical angle.

2. Given two fixed tangents to a variable parabola, and a fixed point through which the axis always passes; prove that the locus of the focus will be a rectangular hyperbola, having its centre at the middle point of the line joining the fixed point to the intersection of the fixed lines, and its asymptotes parallel to the bisectors of the angle between those lines.

I. *Solution by* J. J. Walker, M.A.

1. In the several cases specified the difference of base angles is given. Take the middle point of base ($=2a$, suppose) as origin, and as axes of x and y a line making with the base an angle (θ) equal to half the difference of base angles and a perpendicular thereto (which lines are parallel to the bisectors of the vertical angle); then, for the locus of the vertex, since the angles which the sides make with the axis of x are supplementary,

we have $$\frac{y-a\sin\theta}{x-a\cos\theta}+\frac{y+a\sin\theta}{x+a\cos\theta}=0, \quad \text{or} \quad 2xy=a^2\sin 2\theta.$$

2. Let the points of contact of the tangents be P, P'; T their point of intersection; S the focus; Q the fixed point on the axis; and let the tangents PT, P'T meet the axis in Y, Y'. Then by known properties of the parabola, we have $\angle P'TS - \angle PYS = 0$;

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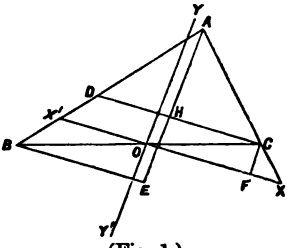
therefore $\angle P'TS + \angle QTY' - (\angle PYS + \angle QTY) = \angle QTY' - \angle QTY$,
that is, $\angle QTS - \angle TQS = \angle QTY' - \angle QTY$;
hence by the former part the locus of S is a rectangular hyperbola.

II. *Solution by the* PROPOSER.

1. Let BC be the given base, O its middle point, and A the vertex whose locus is required. Make AD = AC; join CD; draw BE, XOX′ parallel to CD; and AE, YOY′, CF perpendicular to XOX′. Then the angle BCD or COX is equal to $\frac{1}{2}(C-B)$; hence, taking all as in Mr. WALKER's solution, we have

$$\tan AX'O = \tan ABE = \frac{y + a\sin\theta}{x + a\cos\theta},$$

$$\tan AXO = \tan ACH = \frac{y - a\sin\theta}{a\cos\theta - x};$$

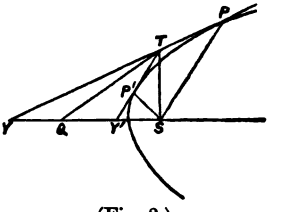


(Fig. 1.)

but $\angle AXO = \angle AX'O$; hence the equation follows as in the former solution.

A *geometrical* investigation of this locus is given in the solutions of my Question 1470, on p. 55 of Vol. I. of the *Reprint*.

2. The accompanying figure will serve to illustrate the manner in which the second part of the Question may be made to depend upon the first, just as given in Mr. WALKER's solution.



(Fig. 2.)

4179. (Proposed by J. J. WALKER, M.A.)—A fixed ray of light falls upon a plane mirror turning about an axis in its plane; show that the reflected rays will generate a ruled quadric surface.

I. *Quaternion Solution by* W. S. McCAY, M.A.

Take the origin on the incident ray (α) at the point where the shortest distance (γ) between it and the axis of rotation (β) meets the former. Then, if θ be the angle between α and β, $\gamma = mV.\beta\alpha$, $T\gamma = m\sin\theta = d$, α and β being unit vectors.

Let μ be the vector normal to the mirror, ρ the vector to any point of the reflected ray.

Then we have $\rho = x\alpha + y\mu\alpha\mu^{-1} = (x-y)\alpha - 2yU\mu S.\alpha U\mu$.

Also $S.\beta\mu = 0$, and $S.(\gamma - x\alpha)\mu = 0$.

For β, the axis, is in the plane of the mirror; and so is $\gamma - x\alpha$, the vector joining the foot of perpendicular on axis to intersection of ray with mirror,

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therefore μ is parallel to $\mathrm{V}.\beta(\gamma - x\alpha)$, and $\mathrm{U}\mu = \dfrac{\mathrm{V}\beta\gamma - x\mathrm{V}\beta\alpha}{\sin\theta\,(m^2+x^2)^{\frac{1}{2}}}$,

observing that $\mathrm{S}.\mathrm{V}\beta\gamma\mathrm{V}\beta\alpha = m\mathrm{S}.\gamma\mathrm{V}\beta\gamma = 0$.

Hence $\mathrm{S}.\alpha\mathrm{U}\mu = \dfrac{\mathrm{S}.\alpha\mathrm{V}\beta\gamma}{\sin\theta\,(m^2+x^2)^{\frac{1}{2}}} = \dfrac{m\mathrm{S}.\alpha\mathrm{V}.\beta\mathrm{V}\beta\alpha}{\sin\theta\,(m^2+x^2)^{\frac{1}{2}}} = \dfrac{m\sin\theta}{(m^2+x^2)^{\frac{1}{2}}}$;

therefore
$$\rho = (x-y)\,\alpha - \frac{2my}{m^2+x^2}(\mathrm{V}.\beta\gamma - x\mathrm{V}.\beta\alpha)$$
$$= (x-y)\,\alpha - \frac{2my\sin\theta}{m^2+x^2}(m\mathrm{UV}.\beta\gamma - x\mathrm{UV}.\beta\alpha).$$

And the corresponding Cartesian equation referred to these vectors as axes, is
$$\mathrm{Y}^2 + \mathrm{Z}^2 = 2(\mathrm{XY}\sin\theta + d\mathrm{Z})$$
a quadric, and evidently ruled, as it should be.

II. *Solution by the* Proposer.

The equation of the mirror, referred to the fixed ray (R) as axis of z, the common perpendicular ($=b$) to it and the line (L) about which the mirror revolves as axis of y, and a third rectangular axis as that of x, will be
$$ax + \lambda(y-b) + cz = 0,$$
λ being a variable parameter and $\dfrac{a}{c}$ being the tangent of the angle which R makes with the plane of L and their common perpendicular.

The direction-cosines of the normal (N) to the mirror are therefore
$$\cos\alpha = \frac{a}{(c^2+a^2+\lambda^2)^{\frac{1}{2}}},\quad \cos\beta = \frac{\lambda}{(c^2+a^2+\lambda^2)^{\frac{1}{2}}},\quad \cos\gamma = \frac{c}{(c^2+a^2+\lambda^2)^{\frac{1}{2}}},$$
while those of the line (P), which is perpendicular both to R and N, are proportional to $-\lambda$, a, 0.

If the direction-cosines of the reflected ray are l, m, n,
$$-\lambda l + am = 0,\quad n = \cos 2\gamma = 2\cos^2\gamma - 1 = \frac{c^2-a^2-\lambda^2}{c^2+a^2+\lambda^2},\quad l^2+m^2+n^2 = 1,$$
the first of these three equations being the condition that the reflected ray should lie in the plane of incidence; *i.e.*, be perpendicular to the line P. Hence $l = \dfrac{2ca}{c^2+a^2+\lambda^2}$, $m = \dfrac{\lambda l}{a}$. The coordinates of the point of incidence are $x'=0$, $y'=0$, $z' = \dfrac{b\lambda}{c}$, and the equations to the reflected ray are $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{cz-b\lambda}{cn}$, in which, if the values of l, m, n given above are substituted, and λ eliminated, the locus of the reflected ray is found to be
$$(a^2-c^2)x^2 + a^2y^2 + 2cazx - 2a^2by = 0.$$
If this surface be referred to the planes of its three axes, viz., the line L, the common perpendicular to it and R, and a line perpendicular to both

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these, its equation is

$$a^2(x^2+y^2)-c^2z^2=a^2b^2,$$

an hyperboloid of one sheet of revolution about the axis of z; viz., the line about which the mirror rotates.

4199. (Proposed by Professor CLIFFORD, M.A.)—Three ternary quadrics U, V, W break up into linear factors 1, 1′; 2, 2′; 3, 3′ respectively. Prove that

$$\square(U, V, W) \equiv 123 \cdot 1'2'3' + 1'23 \cdot 12'3' + 12'3 \cdot 1'23' + 123' \cdot 1'2'3',$$

where $\square(U, V, W)$ is the coefficient of $6\lambda\mu\nu$ in the discriminant of $\lambda U + \mu V + \nu W$, and 123 means the determinant formed with the coefficients of the linear factors 1, 2, 3. Required developments and interpretations.

Solution by Professor WOLSTENHOLME, M.A.

Taking one of each of the linear factors of U, V, W for X, Y, Z, we may write $U \equiv x(x+2cy+2b'z)$, $V \equiv y(2c'x+y+2az)$, $W \equiv z(2bx+2a'y+z)$, whence in the discriminant of $\lambda U + \mu V + \nu W$ the coefficient of $\lambda\mu\nu$ is

$$1-2aa'-2bb'-2cc'+2abc+2a'b'c';$$

also (123) is $\begin{vmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix}$ or 1,

$(1'2'3')$ is $\begin{vmatrix} 1, & 2c, & 2b' \\ 2c', & 1, & 2a \\ 2b, & 2a, & 1 \end{vmatrix}$ or $1-4aa'-4bb'-4cc'+8abc+8a'b'c'$,

$(1'23) \equiv \begin{vmatrix} 1, & 2c, & 2b' \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix} \equiv 1$; $(12'3') \equiv \begin{vmatrix} 1, & 0, & 0 \\ 2c', & 1, & 2a \\ 2b, & 2a', & 1 \end{vmatrix}$ or $1-4aa'$,

and similarly for the others, whence

$$(123)(1'2'3') + (1'23)(12'3') + \ldots$$
$$\equiv 1-4aa'-4bb'-4cc'+8abc+8a'b'c'+1-4aa'+1-4bb'+1-4cc'$$
$$\equiv 4(1-2aa'-2bb'-2cc'+2abc+2a'b'c'),$$

or is to a numerical factor the coefficient of $\lambda\mu\nu$ in the discriminant of $\lambda U + \mu V + \nu W$.

4081. (Proposed by S. BILLS.)—Solve the two following equations:—

$$(x-6)^{\frac{1}{2}} + (x-3)^{\frac{1}{2}} + (x-2)^{\frac{1}{2}} + (x+9)^{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$(x^2-x+3)^{\frac{1}{2}} + (x^2-3x+1)^{\frac{1}{2}} + (x^2+3x-2)^{\frac{1}{2}} + (x^2-x-2)^{\frac{1}{2}} = 10 \ldots\ldots(2).$$

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I. *Solution by* A. M. NASH; Rev. J. L. KITCHIN, M.A.; *and others.*

1. This equation may be written in the form

$$\{(x-6)^{\frac{1}{2}}-1\}+\{(x-3)^{\frac{1}{2}}-2\}+\{(x+z)^{\frac{1}{2}}-3\}+\{(x+9)^{\frac{1}{2}}-4\}=0.$$

Equating each of the expressions in brackets to zero, we find the same value of x, viz., $x=7$; therefore this is a root of the equation.

2. Writing the equation in the form

$$\{(x^2-3x+1)^{\frac{1}{2}}-1\}+\{(x^2-x-2)^{\frac{1}{2}}-2\}+\{(x^2-x+3)^{\frac{1}{2}}-3\}$$
$$+\{(x^2+3x-2)^{\frac{1}{2}}-4\}=0,$$

we see, as before, that each of the expressions in brackets vanishes for the same value $x=3$; this therefore is a root of the equation.

II. *Solution by the* PROPOSER.

1. Put $(x-6)^{\frac{1}{2}}+(x-3)^{\frac{1}{2}}=m$, then $(x-6)^{\frac{1}{2}}=m-(x-3)^{\frac{1}{2}}$; and squaring twice and reducing we obtain

$$2m(x-3)^{\frac{1}{2}}=m^2+3, \text{ and } x=\frac{m^4+18m^2+9}{4m^2} \quad \ldots\ldots\ldots (\alpha).$$

Again, if we assume $(x+2)^{\frac{1}{2}}+(x+9)^{\frac{1}{2}}=n$, we still have, in like manner,

$$x=\frac{n^4-22n^2+49}{4n^2} \quad \ldots\ldots\ldots (\beta).$$

Comparing (α) and (β), and then putting $n=10-m$, as it is by (1), we obtain $n^2-m^2+\frac{49}{n^2}-\frac{9}{m^2}=40$, or $60-20m+\frac{49}{(10-m)^2}-\frac{9}{m^2}=0$,

which reduces to $$(3-m)+\frac{(m-3)(2m+15)}{m^2(10-m)^2}=0 \quad \ldots\ldots\ldots (\gamma).$$

One root of (γ) is $m=3$, whence $x=7$. The equation (γ), when developed, will be of the fifth degree.

Otherwise:—Assume $x+2=y^2$; then $x=y^2-2$, and (1) becomes

$$(y^2-8)^{\frac{1}{2}}+(y^2-5)^{\frac{1}{2}}+(y^2+7)^{\frac{1}{2}}=10-y;$$

and the radicals being removed in the usual manner from this equation, —there being no difficulty when they do not exceed three,—we shall obtain an equation in the fifth degree in y, one root of which will be $y=3$, which gives $x=7$, as before.

2. Assume $(x^2-x+3)^{\frac{1}{2}}+(x^2-3x+1)^{\frac{1}{2}}=m$; then, by removing the radicals as before, we obtain

$$\left(\frac{m^2-1}{m^2}\right)x^2-2\left(\frac{m^2-1}{m^2}\right)x-\left(\frac{m^2}{4}-2+\frac{1}{m^2}\right)=0 \quad \ldots\ldots (\delta).$$

In like manner, if we assume $(x^2-x-2)^{\frac{1}{2}}+(x^2+3x-2)^{\frac{1}{2}}=n$, we shall find

$$\left(\frac{n^2-4}{n^2}\right)x^2+x-\frac{n^2+8}{4}=0,$$

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or, since $n = 10 - m$,

$$\left(\frac{m^2-20m+96}{(10-m)^2}\right) x^2 + x - \tfrac{1}{4}(m^2-20m+108) = 0 \quad \ldots\ldots\ldots (\epsilon).$$

Assume $\dfrac{m^2-1}{m^2} = a$, $\quad 2\left(\dfrac{m^2+1}{m^2}\right) = -b$, $\quad \dfrac{m^2}{4} - 2 + \dfrac{1}{m^2} = -c$,

$$\frac{m^2-20m+96}{(10-m)^2} = a_1, \quad 1 = b_1, \quad \tfrac{1}{4}(m^2-20m+108) = c_1;$$

then (δ) and (ϵ) will become

$$ax^2 + bx + c = 0, \quad a_1x^2 + b_1x + c_1 = 0 \quad \ldots\ldots\ldots (\xi, \eta).$$

Eliminating x between (ξ), (η) we have

$$(ac_1 - a_1c)^2 = (ab_1 - a_1b)(bc_1 - b_1c) \quad \ldots\ldots\ldots (\theta).$$

Restoring the values of a, b, c, &c., we shall obtain from (θ) an equation in m, one root of which will be $m = 4$. Substituting this in the equations (δ) and (ϵ), these become

$$x^2 - \tfrac{24}{15}x = \tfrac{1}{5}, \quad a^2 + \tfrac{8}{3}x = \tfrac{29}{3} \quad \ldots\ldots\ldots (\delta', \epsilon').$$

From (δ') we find $x = 3$ or $-\frac{1}{15}$, and from (ϵ') we find $x = 3$ or $-\frac{37}{8}$; and since $x = 3$ satisfies both these equations, it is the answer required.

4309. (Proposed by R. Tucker, M.A.)—Solve the equations

$$-x^2 + y^2 + z^2 - xy + yz - xz = a = 13 \quad \ldots\ldots\ldots (1),$$
$$x^2 - y^2 + z^2 - xy - yz + xz = b = 1 \quad \ldots\ldots\ldots (2),$$
$$x^2 + y^2 - z^2 + xy - yz - xz = c = -11 \quad \ldots\ldots\ldots (3).$$

I. *Solution by the* Rev. Dr. Booth, F.R.S.

Adding the equations together we have

$$x^2 + y^2 + z^2 - yz - xz - xy = a + b + c \quad \ldots\ldots\ldots (4).$$

Subtracting successively (1), (2), (3) from (4) we have

$$yz = x^2 - \tfrac{1}{2}(b + c), \quad zx = y^2 - \tfrac{1}{2}(a + c), \quad xy = z^2 - \tfrac{1}{2}(a + b) \quad \ldots\ldots (5).$$

Assume $\qquad x + y + z = 3t$, and $a + b + c = 3s \quad \ldots\ldots\ldots (6, 7)$;

then squaring (5) and subtracting (4) from it, we have

$$3t^2 - s = yz + zx + xy = yz + x(y + z) = x^2 - \tfrac{1}{2}(3s - a) + x(3t - x);$$

whence $\quad x = t + \dfrac{s-a}{6t}$; $\quad$ also $\quad y = t + \dfrac{s-b}{6t}$, $\quad$ and $\quad z = t + \dfrac{s-c}{6t}$.

Substituting these values of x, y, z in one of the equations (5), we find

$$6t = \frac{(a^2 + b^2 + c^2 - bc - ca - ab)^{\frac{1}{2}}}{(a + b + c)^{\frac{1}{2}}} = \frac{(a^3 + b^3 + c^3 - 3abc)^{\frac{1}{2}}}{a + b + c};$$

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and thence $x = \dfrac{(b^2+bc+c^2)-a(a+b+c)}{2(a^3+b^3+c^3-3abc)^{\frac{1}{2}}}$, $y =$ &c., $z =$ &c.

With the numerical values given in the question we find $t = \pm 2$, and thence $x = \pm 1$, $y = \pm 2$, $z = \pm 3$.

II. *Solution by the* PROPOSER; S. BILLS; *and others.*

Taking the differences of each pair of the equations (1), (2), (3), we get

$x^2-yz = \frac{1}{2}(b+c)$, $y^2-xz = \frac{1}{2}(a+c)$, $z^2-xy = \frac{1}{2}(a+b)$ ... (i., ii., iii.),

hence, multiplying (i.) by x, (ii.) by y, (iii.) by z, and adding, we get

$$(b+c)x+(c+a)y+(a+b)z = 2(x+y+z)(a^2+y^2+z^2-xy-yz-zx)$$
$$= 2(a+b+c)(x+y+z),$$

whence $(2a+b+c)x+(a+2b+c)y+(a+b+2c)z = 0$ ..... (A).

Again, taking the differences of each pair of equations (i.), (ii.), (iii.), we have $(x-y)(x+y+z) = \frac{1}{2}(b-a)$, &c., whence

$\dfrac{x-y}{y-z} = \dfrac{b-a}{c-b}$; that is, $(b-c)x+(c-a)y+(a-b)z = 0$ ..... (B).

From (A) and (B) we have

$$\frac{x}{-a^2+b^2+c^2-ab+bc-ca} = \frac{y}{a^2-b^2+c^2-ab-bc+ab} = \frac{z}{a^2+b^2-c^2+ab-bc-ca},$$

and substituting in (1) we get

$$x = \frac{(-a^2+b^2+c^2-ab+bc-ca)}{2(a^3+b^3+c^3-3abc)^{\frac{1}{2}}},$$

and similar expressions for y and z. In numbers, we have $x=1$, $y=2$, $z=3$.

III. *Solution by* C. LEUDESDORF.

By adding the equations two and two, they reduce to

$yz-x^2 = \alpha$, $zx-y^2 = \beta$, $xy-z^2 = \gamma$, where $-2\alpha \equiv b+c$, &c.;

or if $$D = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = Xx+Yy+Zz,$$

to $X=\alpha$, $Y=\beta$, $Z=\gamma$. Hence, if we form the determinant $\delta = A\alpha+B\beta+C\gamma$ similarly to D, but using α, β, γ in place of x, y, z, δ will be the reciprocal of D; and $A = xD = \pm x\delta^{\frac{1}{2}}$, &c., &c.;

therefore $$\frac{x}{A} = \frac{y}{B} = \frac{z}{C} = \pm\frac{1}{\delta^{\frac{1}{2}}}.$$

In the particular example given, we have $\alpha=5$, $\beta=-1$, $\gamma=7$;

therefore $\delta = (18)^2$ and $x = \pm 1$, $y = \pm 2$, $z = \pm 3$.

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IV. *Solution by* S. FORDE; A. RENSHAW; *and others.*

Assume $x+y+z=u$, $x^2+y^2+z^2+yz+zx+xy=v$..... (1),
then the equations become $f_1(x, y, z) = v-2xu = a$ ..... (2),
and two similar equations.

By addition, we obtain $3v-2u^2 = a+b+c$;
and from (1) $2v-u^2 = x^2+y^2+z^2 = \frac{1}{4}(v-a)^2+(v-b)^2+(v-c)^2$, by (2);
therefore, eliminating u^2, we have

$$\{3v-(a+b+c)\}\{v+(a+b+c)\} = (v-a)^2+(v-b)^2+(v-c)^2,$$

whence $2(a+b+c)v = a^2+b^2+c^2+ab+bc+ca$;
therefore $2(a+b+c)u^2 = a^2+b^2+c^2-ab-bc-ca.$

But by (2) $$\frac{x}{v-a} = \frac{y}{v-b} = \frac{z}{v-c} = \frac{1}{2u};$$

therefore $$\frac{x}{f_1(a, b, c)} = \frac{y}{f_2(a, b, c)} = \frac{z}{f_3(a, b, c)} = \frac{1}{2(a^2+b^2+c^2-3abc)^{\frac{1}{2}}};$$

or, writing $2a$ for $b+c$, &c., &c.,

$$\frac{x}{a^2-\beta\gamma} = \frac{y}{\beta^2-\gamma a} = \frac{z}{\gamma^2-a\beta} = \frac{1}{(a^3+\beta^3+\gamma^3-3a\beta\gamma)^{\frac{1}{2}}}.$$

In the example given, $a=-5$, $\beta=1$, $\gamma=7$; therefore $x=1$, $y=2$, $z=3$.

4094. (Proposed by Professor TOWNSEND, F.R.S.)—Show that the cuspidal edge of any developable circumscribed to an umbilical quadric projects stereographically from an umbilic upon any parallel to its tangent plane into the evolute of the projection of its curve of contact with the quadric.

Solution by the PROPOSER.

This follows immediately from the known property that any plane section of the quadric projects stereographically from an umbilic upon any parallel to its tangent plane into a circle whose centre is the projection of the vertex of the cone which envelopes the surface along the section; the point of intersection of any three (and therefore of any three consecutive) tangent planes to the quadric projecting consequently into the centre of the circle determined by the projections of their three points of contact with the surface; and therefore, &c.

4197. (Proposed by Professor WOLSTENHOLME, M.A.)—A triangular frame ABC without weight is kept in equilibrium by three forces at right angles to the sides whose lines of action meet in a point L. O is the centre

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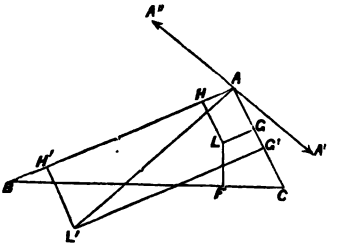
of the circle ABC, LO is produced to L′ so that OL′ = LO; prove that the stresses at A, B, C will be perpendicular and proportional to L′A, L′B, L′C.

Solution by JOHN C. MALET, M.A.

From L draw the perpendiculars LF, LG, LH, and from L′ draw the perpendiculars L′H′ and L′G′, on the sides of the frame. Now if P, Q, R be the three forces, since they act perpendicularly to the sides of the triangle they must be proportional to them; therefore

$$P : Q : R = BC : CA : AB \ldots (1.)$$



Let the stress at A = X, and let it act along the line A″AA′; then for the equilibrium of the bar AC, taking moments about C, we have

$$X . AC . \sin A'AC = Q . GC$$

and for AB,

$$X . AB . \sin A''AC = R . HB,$$

from which by (1) we have

$$\frac{\sin A'AC}{\sin A''AC} = \frac{GC}{HB} = \frac{G'A}{H'A} = \frac{\cos CAL'}{\cos BAL'},$$

therefore AA′ must be perpendicular to AL′; similarly the stresses at B and C are perpendicular to BL′, CL′, which proves the first part of the question.

Let Y and Z be the stresses at B and C. We have now the bar AB in equilibrium under the action of three forces X, Y, R perpendicular to the sides of the triangle AL′B; therefore they must be proportional to the sides, or X : Y = AL′ : BL′. Similarly Y : Z = BL′ : CL′. Therefore X : Y : Z = AL′ : BL′ : CL′, and the theorem is completely proved.

4066. (Proposed by Professor CROFTON, F.R.S.)—A straight line drawn at random crosses a circle; if two points are taken at random within the circle, show that the chance that they fall on opposite sides of the line is $\frac{128}{45\pi^2}$.

I. *Solution by* E. B. ELLIOTT.

If the chord be at distance x from the centre the areas on each side of it are respectively

$$\tfrac{1}{2}\pi a^2 + x(a^2 - x^2)^{\frac{1}{2}} + a^2 \sin^{-1}\frac{x}{a}, \text{ and } \tfrac{1}{2}\pi a^2 - x(a^2 - x^2)^{\frac{1}{2}} - a^2 \sin^{-1}\frac{x}{a}.$$

Hence the chance that of two points taken at random in the circle one lies on each side is

$$\frac{2}{\pi^2 a^4}\left\{\frac{\pi^2 a^4}{4} - \left(x(a^2 - x^2)^{\frac{1}{2}} + a^2 \sin^{-1}\frac{x}{a}\right)^2\right\};$$

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therefore the probability required is

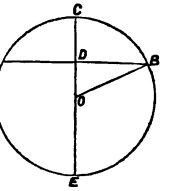
$$\frac{2}{\pi^2 a^4}\int_0^a \left\{\frac{\pi^2 a^4}{4} - \left(x(a^2-x^2)^{\frac{1}{2}} + a^2 \sin^{-1}\frac{x}{a}\right)^2\right\}\frac{dx}{a}$$

$$= \frac{1}{2} - \frac{2}{\pi^2}\int_0^{\frac{1}{4}\pi} (\sin^2\theta\cos^2\theta + 2\theta\sin\theta\cos\theta + \theta^2)\cos\theta\, d\theta$$

$$= \frac{1}{2} - \frac{2}{\pi^2}\left\{\frac{2}{15} + \frac{4}{9} + \frac{\pi^2}{4} - 2\right\} = \frac{128}{45\pi^2}.$$

[Another Solution is given on p. 30 of Vol. XX. of the *Reprint*.]

II. *Solution by* ARTEMAS MARTIN.



Let AB be the random line, $x = OD$, and $r =$ the radius of the circle. Then $AD = BD = (r^2-x^2)^{\frac{1}{2}}$, arc AC = arc $BC = r\cos^{-1}\frac{x}{r}$, area of segment $ADBC = r^2\cos^{-1}\frac{x}{r} - x(r^2-x^2)^{\frac{1}{2}} = S$, area of segment $ADBE = \pi r^2 - S$. The chance that one of the points will fall on S and the other on $\pi r^2 - S$ is

$$\frac{S}{\pi r^2}\times\frac{\pi r^2 - S}{\pi r^2}\times 2 = \frac{2S}{\pi r^2} - \frac{S^2}{\pi^2 r^4},$$

doubling because either point may fall on either side of the line.

Hence $$p = \frac{\int_0^r 2S(\pi r^2 - S)\,dx}{\int_0^r \pi^2 r^4\,dx} = \frac{2}{\pi r^3}\int_0^r S\,dx - \frac{2}{\pi^2 r^5}\int_0^r S\,dx.$$

Now $$\int S\,dx = Sx - \int x\,dS, \text{ and } dS = -2(r^2-x^2)^{\frac{1}{2}}\,dx;$$

therefore $$\int S\,dx = Sx + \int 2x(r^2-x^2)^{\frac{1}{2}}\,dx = Sx - \tfrac{2}{3}(r^2-x^2)^{\frac{3}{2}},$$

$$= \tfrac{2}{3}r^3, \text{ between the limits } x = r,\ x = 0;$$

therefore $$p = \frac{4}{3\pi} - \frac{2}{\pi^2 r^5}\int_0^r S^2\,dx, \quad = \frac{4}{3\pi} - \frac{2S^2 x}{\pi^2 r^5} + \frac{4}{\pi^2 r^5}\int_0^r xS\,dS,$$

since $$\int S^2\,dx = S^2 x - 2\int xS\,dS.$$

Again, $$2\int xS\,dS = -4\int r^2 x(r^2-x^2)^{\frac{1}{2}}\cos^{-1}\left(\frac{x}{r}\right)dx + 4\int x^2(r^2-x^2)\,dx$$

$$= \tfrac{4}{3}r^2(r^2-x^2)^{\frac{3}{2}}\cos^{-1}\left(\frac{x}{r}\right) + \tfrac{4}{3}r^4 x + \tfrac{8}{3}r^2 x^3 - \tfrac{4}{5}x^5;$$

therefore $$\int_0^r S^2\,dx = \frac{2\pi r^5}{3} - \frac{64r^5}{45};$$

therefore $$p = \frac{4}{3\pi} - \frac{2}{\pi^2 r^5}\left(\frac{2\pi r^5}{3} - \frac{64r^5}{45}\right) = \frac{128}{45\pi^2}.$$

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3325. (Proposed by A. B. EVANS, M.A.)—Find general values of x, y, z such that, when a is greater than unity,

$$x^2+axy+y^2=\square,\quad x^2+axz+z^2=\square,\quad y^2+ayz+z^2=\square.$$

I. *Solution by* ARTEMAS MARTIN.

Put $x^2+axy+y^2=(mx-y)^2$; then $\frac{x}{y}=\frac{a+2m}{m^2-1}$. Hence we may take $x=a+2m,\ y=m^2-1$.

Substituting in (2) and (3), we have

$$(a+2m)^2+a(a+2m)z+z^2=\square,\quad (m^2-1)^2+a(m^2-1)z+z^2=\square\ldots(4,5).$$

Putting $(5)=[nz-(m^2-1)]^2$, we find $z=\frac{(a+2n)(m^2-1)}{n^2-1}=\frac{m^2-1}{b}$, by putting $b=\frac{n^2-1}{a+2n}$.

Substituting in (4), we have, after proper reduction,

$$m^4+2abm^3+(a^2b+4b^2-2)m^2+2ab(2b-1)m+a^2b^2-a^2b+1=\square\ldots\ldots(6).$$

Put $(6)=[m^2+abm+\frac{1}{2}(a^2b+4b^2-a^2b^2-2)]^2$, and we get

$$m=\frac{\frac{1}{4}(a^2b+4b^2-a^2b^2-2)^2+a^2b-a^2b^2-1}{ab^2(a^2b-a^2-4b+4)}.$$

II. *Solution by the* PROPOSER.

Put $x^2+axy+y^2=\left(x-\frac{m}{n}y\right)^2$. Reducing we have $x=\frac{(m^2-n^2)y}{2mn+an^2}$, where x may be taken $=m^2-n^2$.....(1), and $y=2mn+an^2$.....(2).

Considering x, y as given quantities, we have to make

$$x^2+axz+z^2=\square=B^2,\text{ and }y^2+ayz+z^2=\square=A^2.$$

By subtraction we have

$$y^2-x^2+a(y-x)z=(y-x)(y+x+az)$$

$$=\tfrac{1}{2}a(y-x)\times 2\left(\frac{y+x}{a}+z\right)=A^2-B^2=(A+B)(A-B).$$

Put $\quad \tfrac{1}{2}a(y-x)=A-B,\text{ and }2\left(\frac{y+x}{a}+z\right)=A+B,$

then we shall have $\quad A=\frac{a^2+4}{4a}y-\frac{a^2-4}{4a}x+z.$

By substitution we have $\quad y^2+ayz+z^2=A^2=\left(\frac{a^2+4}{4a}y-\frac{a^2-4}{4a}+z\right)^2.$

Reducing we shall find

$$z=\left\{\left(\frac{a^2+4}{4a}y-\frac{a^2-4}{4a}x\right)^2-y^2\right\}\div\frac{a^2-4}{2a}(y+x)$$

$$=\left\{\left\{(a^2+4)y-(a^2-4)x\right\}^2-16a^2y^2\right\}\div 8a(a^2-4)(y+x)$$

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which can be reduced to

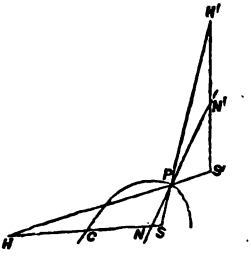
$$z = \{(a+2)y-(a-2)x\} \times \{(a-2)y-(a+2)x\} \div 8a(y+x) \ldots\ldots (3).$$

We have then (1), (2), (3) for general values of x, y, z, where a may be any number > 1, and m, n any number that will make x and z positive. If $a=2$, then $z=\frac{0}{8}$, *i.e.* any number whatever; x, y can also be so taken. If $a=3$, $m=4$, $n=3$, then $x=7$, $y=51$, $z=\frac{14.8}{17}$; or in integers, $x=609$, $y=4437$, $z=248$. If $a=4$, $m=5$, $n=3$, then $x=16$, $y=66$, $z=\frac{3.7.8}{4.1}$; or in integers, $a=656$, $y=2706$, $z=273$.

4275. (Proposed by W. H. H. Hudson, M.A.)—If S, H be the poles of a Lemniscate, P any point on it, C the middle point of SH, and if PH′, PS′ be taken on PS, PH, equal respectively to PH, PS; prove that the normal at P bisects S′H′.

I. *Solution by* E. B. Elliott.

Take SP and HP as coordinates r and r'; then the equation of the Lemniscate is $rr' = c^2$. Now if ϕ and ϕ' be the angles SPN, HPN made with the normal by r and r' it is easily seen, by taking two consecutive points on the curve, that $\frac{dr}{ds} = \sin\phi$, $-\frac{dr'}{ds} = \sin\phi$.



But from the equation of the curve

$r'dr + rdr' = 0$, therefore $r'\sin\phi = r\sin\phi'$;

i.e. $PH'\sin H'PN' = PS'\sin S'PN'$;

therefore NPN′, the normal at P, bisects S′H′.

This proof applies to any Cassinian oval as well as to the Lemniscate.

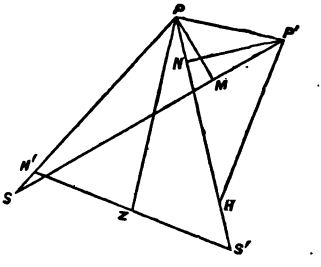
II. *Solution by* C. Leudesdorf.

Take a point P′ on the curve consecutive to P; draw PM, P′N perpendicular from P, P′ respectively on SP′ and HP; join PP′ and draw PZ perpendicular to PP′, and cutting S′H′ in Z. Then

$$PS.PH = P'S.P'H$$
$$= (PS+P'M)(PH-PN);$$

therefore, neglecting infinitesimals of the second order,

$$PS.PN = PH.P'M.$$



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But $PN = PP' \cos SPP' = PP' \sin ZPS'$

$P'M = PP' \sin MPP' = PP' \sin (ZPH' + PSP')$

$= PP' \sin ZPH'$ ultimately,

and $PS = PS', \quad PH = PH';$

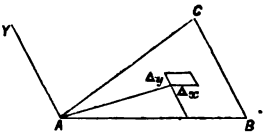
$\therefore PS' \sin ZPS' = PH' \sin ZPH', \quad \therefore \Delta PZH' = \Delta PZS', \quad \therefore ZH' = ZS';$

or PZ, which is ultimately the normal at P, bisects S'H'.

2032. (Proposed by N'IMPORTE.)—To investigate the expression for the moment of inertia of a triangle about one of its angular points.

Solution by the PROPOSER.

Let ABC be the triangle; A the centre of gyration. Take A as origin of coordinates; AB as axis of x, and AY parallel to BC as axis of y; then the equation of AC is $y = \frac{\sin A}{\sin C} x = \frac{a}{c} x.$



Let S = area of triangle, m = mass of a unit of area, (x, y) any point in the triangle, Δx, Δy the sides of an elementary parallelogram.

Now mass of element $= m \Delta x \, . \, \Delta y \sin B$; therefore its moment of inertia about A is $m (x^2 + y^2 - 2xy \cos B) \Delta x \Delta y$;

hence the moment of inertia of the whole triangle about A

$$= m \sin B \int_0^c \int_0^{\frac{ax}{c}} (x^2 + y^2 - 2xy \cos B) \, dx \, dy$$

$$= m \sin B \int_0^c \left(\frac{ax^3}{c} + \frac{a^3 x^3}{3c^3} - \frac{a^2 x^3}{c^2} \cos B \right) dx$$

$$= ma \sin B \left(\tfrac{1}{4} c^3 + \tfrac{1}{12} a^2 c - \tfrac{1}{4} ac^2 \cos B \right)$$

$$= \tfrac{1}{12} mac \sin B \left\{ 3c^2 + a^2 - \tfrac{3}{2} (a^2 + c^2 - b^2) \right\}$$

$$= \tfrac{1}{12} mS \left\{ 3(b^2 + c^2) - a^2 \right\} = \tfrac{1}{12} \left\{ 3 (b^2 + c^2) - a^2 \right\} \text{ mass of triangle.}$$

4101. (Proposed by J. J. WALKER, M.A.)—If O be the centre of the circle circumscribing the plane triangle ABC, and AO meet the base BC in A', and O' be taken harmonic conjugate to O with respect to AA'; if also O'' and O''' be analogous points in reference to B and C respectively; prove

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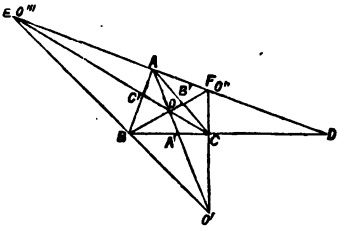
(1) that the triangle O′O″O‴ circumscribes ABC; and (2) that

$$OO' : OO'' : OO''' = \cos^2 A : \cos^2 B : \cos^2 C.$$

Solution by ASHER B. EVANS, M.A.



1. Through O′, the harmonic conjugate to O with respect to AA′, draw O′C and O′B cutting BO and CO produced in F and E respectively. Since each diagonal of a complete quadrilateral is divided harmonically by the other two, O′O, a diagonal of the complete quadrilateral O′BOCEF is divided harmonically by BC and EF. But O′O is divided harmonically in A′ and A; and, moreover, A′ is on BC, therefore A is on EF. By reason of the complete quadrilaterals EAOBFO′ and FAOCEO′, the diagonals EO and FO are divided harmonically in C′ and C, B′ and B respectively. It is now evident that O″ and O‴ coincide with F and E, and that the triangle O′O″O‴ circumscribes ABC.

2. Again; $AO = R$, $A'O = \dfrac{R\cos A}{\cos(C-B)}$ and $AO : A'O = AO' : A'O'$.

Hence $$OO' = \frac{2R\cos A}{\cos(C-B)-\cos A} = \frac{R\cos^2 A}{\cos A\cos B\cos C}.$$

Similarly, $$OO'' = \frac{R\cos^2 B}{\cos A\cos B\cos C} \text{ and } OO''' = \frac{R\cos^2 C}{\cos A\cos B\cos C};$$

and therefore $OO' : OO'' : OO''' = \cos^2 A : \cos^2 B : \cos^2 C$.

4145. (Proposed by J. J. WALKER, M.A.)—Prove that the equation to the circle passing through a point $(x'\ y')$ and through the points of contact of tangents from it to the parabola $y^2 - 4ax = 0$ is

$$a(x^2+y^2) - (2a^2+y'^2)\,x - y'(a-x')\,y + ax'(2a-x') = 0.$$

I. *Solution by* R. W. GENESE, B.A.

The points of contact are determined by the equations

$$y^2 = 4ax, \quad yy' = 2a(x+x') \quad \ldots\ldots\ldots\ldots (1, 2).$$

From (2), we have $$(x+x')^2 = \frac{y'^2y^2}{4a^2} = y'^2\frac{x}{a},$$

therefore $$(x+x')^2 - y'^2\left(\frac{x}{a}\right) + (y^2-4ax) = \lambda\left\{yy' - 2a(x+x')\right\}$$

is the equation to *any* circle through the two points.

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If the circle be required to pass through the point (x', y'), we have

$$4x'^2 - y'^2 \frac{x'}{a} + y'^2 - 4ax' = \lambda (y'^2 - 4ax'); \text{ or, } \lambda = 1 - \frac{x'}{a}.$$

Using this value of λ, we get the required equation.

[The substitution is simplified by re-writing the equation in the form

$$(x + x' + \lambda)^2 + y^2 - \frac{y'^2 + 4a^2}{a} x - \lambda y' y - \lambda^2 = 0 \,]$$

II. *Solution by* R. TUCKER, M.A.

Let m_1, m_2 be the two points on the parabola; then the equations to the tangents and normals are

$$y = m_1 x + \frac{a}{m_1}, \qquad y = m_2 x + \frac{a}{m_2},$$

$$m_1 y + x = 2a + \frac{a}{m_1^2}, \qquad m_1 y + x = 2a + \frac{a}{m_2^2};$$

hence the equation to the circle required is

$$\left(y - m_1 x - \frac{a}{m_1}\right)\left(m_2 y + x - 2a - \frac{a}{m_2^2}\right) = \left(y - m_2 x - \frac{a}{m_2}\right)\left(m_1 y + x - 2a - \frac{a}{m_1^2}\right).$$

Multiplying out, and dividing all through by $m_1 - m_2$, we get

$$x^2 + y^2 - ay\left[\frac{m_1 + m_2}{m_1 m_2}\right]\left[1 - \frac{1}{m_1 m_2}\right] - ax\left[2 + \frac{(m_1 + m_2^2)}{m_1^2 m_2^2}\right] + \frac{a^2}{m_1 m_2}\left[2 - \frac{1}{m_1 m_2}\right] = 0 \ldots\ldots (1);$$

but $$x' = \frac{a}{m_1 m_2}, \quad \text{and} \quad y' = a\left(\frac{1}{m_1} + \frac{1}{m_2}\right),$$

hence (1) may be written

$$a(x^2 + y^2) - yy'(a - x') - x(2a^2 + y'^2) + ax'(2a - x') = 0.$$

4322. (Proposed by Professor CROFTON, F.R.S.)—A heavy uniform beam AB is supported on two props at the points C, D; prove the following construction for finding the two points where the bending moment vanishes. With C, D as centres draw two circles passing respectively through A and B; draw any circle cutting both these circles orthogonally; it will cut the beam AB in the two points of inflexion required. The beam might therefore be jointed at these points. If the circle does not meet AB, there is no such point, and the beam is convex upwards throughout its whole length.

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I. *Solution by* Professor Townsend, F.R.S.

This very elegant, and so far as I am aware novel, construction for the determination of the two points of inflexion X and Y of the strained beam, may be proved immediately as follows:—

Denoting by P and Q the two reactions at C and D arising from its weight W, then since, at any point of it X between C and D, where X and Y if real must manifestly lie, the bending moment M

$$= \frac{1}{2} \cdot \frac{W}{AB} \cdot AX^2 - P \cdot CX = \frac{1}{2} \cdot \frac{W}{AB} \cdot \left((AC + CX)^2 - 2 \cdot \frac{P}{W} \cdot AB \cdot CX \right),$$

we have at once, for the determination of the two points X and Y for which M = 0, that is for the two points of inflexion, real or imaginary, of the beam, the quadratic equation

$$CX^2 + 2\left(AC - \frac{P}{W} \cdot AB\right) CX + CA^2 = 0,$$

the product of whose roots $CX \cdot CY = CA^2$; and, since for the same reason $DX \cdot DY = DB^2$, therefore, &c.; the two points X and Y being (see my *Modern Geometry*, Vol. I., Chap. IX., Art. 152) those which are inverse at once to the two circles of which C and D are the centres, and CA and DB the radii, which are determined most readily by the construction above given.

II. *Solution by* J. C. Malet, M.A.

Take for axes the line CD and a perpendicular to it through the middle point of the bar, the positive direction being towards D. Let $CD = d$, $DB = b$, $AC = c$, $AB = 2a$, then the equations of the circles with C and D as centres and radii equal c and b respectively, are

$$\left\{x + \frac{d+b-c}{2}\right\}^2 + y^2 - c^2 = 0, \quad \left\{x - \frac{d-b+c}{2}\right\}^2 + y^2 - b^2 = 0;$$

and the equation of any circle coaxial with these circles is

$$x^2 + y^2 - x\left\{d\frac{1-\lambda}{1+\lambda} + c - b\right\} + \frac{d^2 + (c-b)^2}{4} + \frac{d(c-b)}{2} \cdot \frac{1-\lambda}{1+\lambda} - \frac{b^2 + c^2\lambda}{1+\lambda} = 0 \quad \ldots\ldots\ldots\ldots (1),$$

λ being an indeterminate quantity.

If X is the ordinate of the centre of this circle, we have

$$2X = d\frac{1-\lambda}{1+\lambda} + c - b,$$

from which we get $\frac{1}{1+\lambda} = \frac{2X - c + b + d}{2d}$, $\frac{\lambda}{1+\lambda} = \frac{d - 2X + c - b}{2d}$;

hence we find, for the condition that radius of circle (1) should vanish,

$$X^2 - \frac{c-b}{2}(2X - c + b) + \frac{2X(b^2 - c^2) + (c-b)(c^2 - b^2) + d(c^2 + b^2)}{2d} - \frac{d^2 + (c-b)^2}{4} = 0,$$

$$dX^2 - 2Xa(c-b) + a\left\{(c-b)^2 + \tfrac{1}{2}d(b + c - d)\right\} = 0 \quad \ldots\ldots\ldots\ldots (2),$$

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which equation determines the positions of the "limiting points" of the system of coaxial circles, through which every circle orthogonal to the system must pass.

Let now P and Q be the reactions at D and C, W the weight of the beam; and if M be the breaking moment downwards at any point of the bar, we have

$$\frac{W}{2}\cdot\frac{(a-x}{2a}+\frac{W}{2}\frac{(a+x)^2}{2a}-P\left\{\frac{d-b+c}{2}-x\right\}-Q\left\{\frac{d+b-c}{2}+x\right\}=M.$$

But from the conditions of equilibrium we have

$$Pd=W\left\{\frac{d+b-c}{2}\right\},\quad Qd=W\left\{\frac{d-b+c}{2}\right\}.$$

Substituting and reducing we find

$$\frac{W}{2ad}\left\{dx^2-2xa(c-b)+a\left\{(c-b)^2+\tfrac{1}{2}d(b+c-d)\right\}\right\}=M\ldots\ldots(3).$$

From this it is seen we have the same equation to determine the points where there is no tendency to break as to determine the points where any circle orthogonal to system (1) cuts the bar; which proves the theorem.

If the expression

$$dx^2-2xa(c-b)+a\left\{(c-b)^2+\tfrac{1}{2}d(b+c-d)\right\}=d(x-\alpha)(x-\beta)=F(x),$$

where α is greater than β, we see that when x is greater than α or less than β, then $\frac{W}{2ad}F(x)$, or M, is positive; hence we see that the bar is concave upwards between the points where the tendency to break vanishes, and outside these points it is convex upwards. If $b+c=d$, or there is as much of the bar outside the props as between them, then the "limiting points" coincide and the bar is everywhere convex upwards; and if $b+c>d$, the limiting points are imaginary and the bar is everywhere convex upwards.

In the first case, where the "limiting points" are real, we have

$$F\left\{\frac{a(c-b)}{d}+x\right\}=F\left\{\frac{a(c-b)}{d}-x\right\};$$

hence in this case the bar is symmetrical with regard to a point distant from the centre of gravity $\frac{a(c-b)}{d}$; in this case, since $F(x)$ increases the more we increase x from α or diminish it from β, it is evident the extremities of the bar are the highest points. From these considerations we can easily see the form the rod will assume in each case, supposing it to be nearly inelastic.

III. *Solution by* G. S. Carr.

Let AC = a, BD = b, CD = c, CX = x, P = reaction of the support at C, W = the weight of the part AX, M = the bending moment at X some point between C and D. Then $M=W.\frac{1}{2}(a+x)-Px$. Also since the beam is uniform we may take $W=a+x$; and by taking moments round D we also have

$$P=\frac{(a+c)^2-b^2}{2c};\ \text{therefore}\ 2M=x^2-\frac{a^2+c^2-b^2}{c}x+a^2.$$

Now let k, k' be the distances of C and D from the radical axis of the

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two circles. Then since $k^2-k'^2 = a^2-b^2$ and $k+k' = c$, we have

$$k = \frac{a^2+c^2-b^2}{2c}; \text{ therefore } 2M = x^2-2kx+a^2.$$

For M to vanish we must have then

$$x = k \pm (k^2-a^2)^{\frac{1}{2}} = k \pm \delta \text{ (SALMON'S } \textit{Conics}\text{, p. 101),}$$

showing that the bending moment vanishes at Poncelet's limiting points; and these points are defined by any circle cutting the two existing circles orthogonally, provided those circles do not cut each other. Hence, if $a+b$ be greater than c, there is no change of curvature in the beam.

4227. (Proposed by the EDITOR.)—Show that (1) the Boothian tangential equation of the evolute of the quadrantal hypocycloid is

$$a^2(\xi^2-\eta^2)^2 = \xi^2+\eta^2;$$

(2) the equation of an involute of the hypocycloid is

$$a^2(\xi^2-\eta^2)^2 = 4(\xi^2+\eta^2)\left\{1-c(\xi^2+\eta^2)^{\frac{1}{2}}\right\}^2.$$

Solution by Professor WOLSTENHOLME, M.A.

1. The tangent to the hypocycloid has a constant length a intercepted between the axes, if the axes be the cuspidal tangents. Hence the normal or tangent to the evolute will have a constant length $(2a)$ cut off by the straight lines bisecting the axes, and its equation will be

$$\frac{x-y}{\sqrt{2}.\cos\theta} + \frac{x+y}{\sqrt{2}.\sin\theta} = 2a,$$

whence $\xi = \frac{1}{2\sqrt{2}.a}\cdot\left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)$; $\eta = \frac{1}{2\sqrt{2}.a}\left(\frac{1}{\sin\theta}-\frac{1}{\cos\theta}\right)$;

therefore $\frac{1}{(\xi+\eta)^2} + \frac{1}{(\xi-\eta)^2} = 2a^2$, or $a^2(\xi^2-\eta^2)^2 = \xi^2+\eta^2$.

2. The equation of any involute of the hypocycloid will be found by taking $\frac{1}{4}a$ instead of a, and writing $\frac{\xi}{1-c(\xi^2+\eta^2)^{\frac{1}{2}}}$, $\frac{\eta}{1-c(\xi^2+\eta^2)^{\frac{1}{2}}}$ for ξ and η respectively; or will be $a^2(\xi^2-\eta^2)^2 = 4(\xi^2+\eta^2)\left\{1-c(\xi^2+\eta^2)^{\frac{1}{2}}\right\}^2$.

4175. (Proposed by Dr. BALL, F.R.S.)—A free rigid body is acted upon by five forces, show how to move the body so that it shall not do work against nor receive energy from any one of the forces.

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Solution by the PROPOSER.

For explanation of the language, reference may be made to the "Theory of Screws," *Transactions of the Royal Irish Academy*, Vol. xxv., p. 157.

Let $A_1, \dots A_5$ be the five forces. Draw two transversals L, M intersecting $A_1, \dots A_4$. Construct the cylindroid of which L, M are the screws of zero pitch; find, upon this cylindroid, the screw X reciprocal to A_5. Then the only movement which the body can receive, so as to fulfil the prescribed conditions, is a twist about the screw X. For X is then reciprocal to $A_1, \dots A_5$, and therefore a body only free to twist about X will be unacted upon by *any* forces directed along $A_1, \dots A_5$.

From the theory of reciprocal screws it follows that a body rotated around any of the lines $A_1, \dots A_5$ will not do work against nor receive energy from a wrench about X.

As a particular case, if $A_1, \dots A_5$ have a common transversal, then X is that transversal, and its pitch is zero. In this case it is sufficiently obvious that $A_1, \dots A_5$ cannot disturb the equilibrium of a body only free to rotate about X.

3803. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—If $f(x, y) \equiv \mathrm{F} = 0$ be the projective equation of a plane curve, show that the tangential equation of its evolute may be found by eliminating x and y from the equations

$$\xi = \frac{d\mathrm{F}}{dy} \div \left(\frac{d\mathrm{F}}{dy}x - \frac{d\mathrm{F}}{dx}y\right), \quad v = -\frac{d\mathrm{F}}{dx} \div \left(\frac{d\mathrm{F}}{dy}x - \frac{d\mathrm{F}}{dx}y\right), \quad f(x, y) = 0;$$

and (2) show therefrom that the tangential equation to the evolute of the ellipse is $a^2v^2 + b^2\xi^2 = (a^2 - b^2)\,\xi^2v^2$.

Solution by J. J. WALKER, M.A.

1. The reciprocals of ξ and v, as defined in the question, are obviously the intercepts on the axes of coordinates made by the normal at (x, y) so that $x\xi + yv = 1$ is its equation. The evolute being the envelope of this line, its tangential equation will evidently be the relation between ξ and v found by eliminating x, y among the three equations given in the question.

2. For the central conic $b^2x^2 + a^2y^2 - a^2b^2 = 0$, we have

$$\frac{1}{\xi} = \frac{(a^2 - b^2)\,x}{a^2} \text{ or } \frac{a^2b}{\xi} = (a^2 - b^2)\,bx; \quad \frac{ab^2}{v} = -(a^2 - b^2)\,ay.$$

Squaring and adding these equations, we obtain

$$\frac{a^4b^2}{\xi^2} + \frac{a^2b^4}{v^2} = (b^2x^2 + a^2y^2)(a^2 - b^2)^2 = a^2b^2\,(a^2 - b^2)^2,$$

or

$$b^2\xi^2 + a^2v^2 = (a^2 - b^2)^2\,\xi^2v^2.$$

3895. (Proposed by J. GRIFFITHS, M.A.)—If F denote a homogeneous function of the degree n, show that the locus of the foci of an inscribed conic

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$(\lambda\alpha)^{\frac{1}{2}} + (\mu\beta)^{\frac{1}{2}} + (\nu\gamma)^{\frac{1}{2}} = 0$, where λ, μ, ν satisfy the relation $F(\lambda, \mu, \nu) = 0$, is an nth circular curve of the degree $3n$ given by the equation $F(X,Y,Z) = 0$; X, Y, Z being, for shortness, respectively written for

$$x(y^2 + z^2 + 2yz\cos A), \quad y(z^2 + x^2 + 2zx\cos B), \quad z(x^2 + y^2 + 2xy\cos C).$$

Solution by R. F. Scott.

The tangential equation of the given conic is

$$\Sigma = \lambda vw + \mu wu + \nu uv = 0;$$

and $\quad \Sigma' = u^2 + v^2 + w^2 - 2vw\cos A - 2wu\cos B - 2uv\cos C = 0,$

is the equation of the circular points.

And, as explained in Salmon's *Conics*, p. 340, if we make

$$\Sigma + k\Sigma' \equiv (xu + yv + zw)(x'u + y'v + z'w),$$

then (x, y, z) and (x', y', z') are foci.

This gives $\quad xx' = yy' = zz' = k, \quad yz' + zy' = \lambda - 2k\cos A,$

and two similar equations; whence

$$kx \quad + z^2 + 2yz\cos A) = \lambda xyz,$$

and two similar equations.

Substituting for λ, μ, ν, we get, for the locus of the foci,

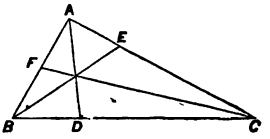
$$F(x, y, z) = 0.$$

4284. (Proposed by J. L. McKenzie.)—If l, m, n be the lines bisecting the angles of the triangle ABC, and terminating in the opposite sides, and Δ the area, prove that

$$\Delta = \tfrac{1}{2}l(b+c)\sin\frac{A}{2} = \tfrac{1}{2}la\cos\tfrac{1}{2}(B-C) = \frac{lmn(b+c)(c+a)(a+b)}{4abc(a+b+c)}.$$

Solution by J. Merrifield, F.R.A.S.; T. T. Wilkinson, F.R.A.S.; *and many others.*

$$\begin{aligned}\Delta &= (\tfrac{1}{2}bl\sin\tfrac{1}{2}A + \tfrac{1}{2}lc\sin\tfrac{1}{2}A)\\ &= (\tfrac{1}{2}l.\mathrm{DC}\sin D + \tfrac{1}{2}l.\mathrm{DB}\sin D)\\ &= \tfrac{1}{2}l(b+c)\sin\tfrac{1}{2}A = \tfrac{1}{2}la\sin D\\ &= \tfrac{1}{2}la\cos\tfrac{1}{2}(B-C).\end{aligned}$$



Using the formula with the other bisectors, we get

$$\Delta^2 = \tfrac{1}{8}lmn(b+c)(c+a)(a+b)\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$

$$= \tfrac{1}{8}lmn(b+c)(c+a)(a+b)\left\{\frac{s_1s_2s_3}{abc} \text{ or } \frac{\Delta^2}{abcs}\right\};$$

therefore

$$\Delta = \frac{lmn(a+b)(b+c)(c+a)}{4abc(a+b+c)}.$$

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4013. (Proposed by G. M. MINCHIN, M.A.)—If $X = (1-\lambda^2x^2)(1-\mu^2x^2)$, prove that

$$\int_0^1 \frac{x^2dx}{(1-\lambda^2x^2)X^{\frac{1}{2}}} + \int_0^1 \frac{x^2dx}{(1-\mu^2x^2)X^{\frac{1}{2}}} + \int_0^1 \frac{x^2dx}{X^{\frac{1}{2}}} = \frac{1}{\{(1-\lambda^2)(1-\mu^2)\}^{\frac{1}{2}}}.$$

Solution by the PROPOSER.

Let $\phi = \int_0^{\frac{a}{m}} \frac{y^2dy}{a\{(a^2-h^2y^2)(a^2-k^2y^2)\}^{\frac{1}{2}}}$; and let $y = \frac{a}{m}x$;

therefore $\phi = \frac{1}{m^3}\int_0^1 \frac{x^2dx}{\left\{\left(1-\frac{h^2}{m^2}x^2\right)\left(1-\frac{k^2}{m^2}x^2\right)\right\}^{\frac{1}{2}}}$;

hence ϕ is independent of a, therefore $\frac{d\phi}{da} = 0$. But we have

$$\frac{d\phi}{da} = \frac{1}{m^3a}\cdot\frac{1}{\left\{\left(1-\frac{h^2}{m^2}\right)\left(1-\frac{k^2}{m^2}\right)\right\}^{\frac{1}{2}}} - \int_0^{\frac{a}{m}} y^2dy\,\frac{d}{da}\cdot\frac{1}{a\{(a^2-h^2y^2)(a^2-k^2y^2)\}^{\frac{1}{2}}}$$

$$= \frac{1}{m^3a\left\{\left(1-\frac{h^2}{m^2}\right)\left(1-\frac{k^2}{m^2}\right)\right\}^{\frac{1}{2}}} - \int_0^{\frac{a}{m}} \frac{y^2dy}{a^2\{(a^2-h^2y^2)(a^2-k^2y^2)\}^{\frac{1}{2}}} \times\left(1+\frac{a^2}{a^2-h^2y^2}+\frac{a^2}{1-k^2y^2}\right).$$

Let y $\frac{a}{m}x$, and $\frac{h}{m} = \lambda$, $\frac{k}{m} = \mu$, and the required result follows.

4233. (Proposed by Professor TOWNSEND, F.R.S.)—Show that every arc of a bifocal conic, described freely under the action of a central force emanating (in the proper direction) from either focus, is brachystochronous, for the velocity of its free description, under the action of an equal force emanating (in the proper direction) from the other focus. State also the general property of brachystochronous compared with free motion, of which the above is a particular case.

I. *Solution by* Professor WOLSTENHOLME, M.A.

In a free path, v being the velocity at any point, $\int v\,ds$ is a minimum; in the brachystochrone $\int \frac{dv}{v}$ is a minimum. Hence since for central forces we know that the former leads to $v \propto p^{-1}$, we shall have in the latter $v^{-1} \propto p^{-1}$ or $v \propto p$, p being the perpendicular from the centre of force on the tangent. But in the ellipse, if p, p' be the perpendiculars from the two foci, pp' = constant, hence in the free path about one focus $v \propto p^{-1} \propto p'$,

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or the path is a brachystochrone (with the same law of velocity) for a force in the other focus. The force will be repulsive in the brachystochrone and attractive in the free path. This example and the general theorem are due to Professor Tait and Sir W. Thomson, (see Tait and Steele's *Dynamics of a Particle*, 3rd ed., pp. 318—320. The general theorem is as follows:—If T be the time through any arc of a given brachystochrone, the same path will be described freely under the action of forces whose potential is V_1 where $2(H_1 - V_1) = \frac{\{\phi'(\tau)\}^2}{2(H - V)}$, ϕ' being any function whatever; and $\phi(\tau)$ will represent the action in the free path. With other notation this amounts to $vv' = \phi'(\tau)$; where v, v' are the velocities in the two paths.

If we have two curves P and Q, of which P is a free path and Q a brachystochrone for a given conservative system of forces; P will be a brachystochrone for a system of forces for which Q is a free path; and the action and time in any arc of either when it is a free path are functions of the time and action respectively in the same arc when it is a brachystochrone. (See *Transactions* of the Royal Society of Edinburgh, 1865.)

II. *Solution by the* PROPOSER.

Denoting by p and p' the two perpendiculars from the two foci F and F' upon the current tangent to the arc; then, since the current velocity of description varies inversely as p for free motion round F, and directly as p' for brachystochronous motion round F', and since in every bifocal conic $pp' =$ a constant; therefore &c., as regards the particular property in question.

The components perpendicular to the direction of motion of the forces of action and inertia being, in all cases, equal and opposite for free, and equal and coincident for brachystochronous motion, under the action of any force having a potential, and that too irrespectively of whether the motion be in three or only in two dimensions, it follows consequently that *every arc of a curve, described freely under the action of any force having a potential, is brachystochronous, for the velocity of its free description, under the action of the reflexion (as regards at once magnitude, direction, and line of action) of the force with respect to the current tangent to the arc.* This is accordingly the general property, of which that of the question is manifestly a very particular case.

4012. (Proposed by Professor TOWNSEND, F.R.S.)—Show that the internal surfaces of equilibrium of a uniform ellipsoidal shell of any thickness bounded by coaxal, but not similar surfaces, for the law of the inverse square of the distance, are in general two systems of coaxal and similar hyperboloids, of one and of two sheets respectively, separated by a cone asymptotic to them all; show also where they degenerate into two corresponding systems of hyperbolic cylinders, separated by a pair of planes asymptotic to them all.

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Solution by the PROPOSER.

Denoting by A, B, C and by A′, B′, C′ the six coefficients of attraction for the two uniform solids bounded by the two surfaces S and S′ respectively, and by X, Y, Z the three components of the attraction of the included shell at any internal point x, y, z; since then, on well known principles,

$$X = (A - A')x, \quad Y = (B - B')y, \quad Z = (C - C')z,$$

the differential equation of the internal system of equilibrium surfaces is consequently $(A - A')x\,dx + (B - B')y\,dy + (C - C')z\,dz = 0$,

which proves the general property, since, as is also well known,

$$A + B + C = A' + B' + C' = 4\pi\rho.$$

When any one of the three relations $A = A'$, $B = B'$, $C = C'$ exists, the hyperboloids of course all degenerate into cylinders.

3790. (Proposed by S. WATSON.)—From a fixed point P on the circumference of a circle, a ball whose elasticity is e is struck so as to impinge on the circle at two other points Q and R, and then return to P. Find the average area of the triangle PQR for all values of e.

Solution by S. FORDE.

Let the angles of incidence at Q, R, P be α, β, γ; then $\alpha + \beta + \gamma = \frac{1}{2}\pi$, and

$$\frac{\cot\gamma}{e^2} = \frac{\cot\beta}{e} = \frac{\cot\alpha}{1} = \left(\frac{1 + e + e^2}{e^3}\right)^{\frac{1}{2}};$$

therefore taking the radius of the circle as unity we have $\Delta = \frac{1}{2}(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)$

$$= \frac{\cot\alpha}{\cot^2\alpha + 1} + \ldots + \ldots$$

$$= \cot\alpha\left[\frac{e^3}{1 + e + e^2 + e^3} + \frac{e^2}{(1 + e)^2} + \frac{e^2}{1 + e + e^2 + e^3}\right]$$

$$= 2\,\frac{e^{\frac{1}{2}}(1 + e + e^2)^{\frac{3}{2}}}{(1 + e^2)(1 + e)^2};$$

therefore Average value of $\Delta = 2\int_0^1 \frac{e^{\frac{1}{2}}(1 + e + e^2)^{\frac{3}{2}}}{(1 + e^2)(1 + e)^2}\,de.$

4231. (Proposed by J. J. SYLVESTER, F.R.S.)—1. If two opposite angles of a jointed rhombus move in the same fixed circle, and one of the other angles in a second fixed circle passing through the centre of the first, prove that the fourth angle will move in a straight line.

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2. Show that it is possible, by means of an apparatus consisting exclusively of rigid rods, compass joints, and pivots, to convert circular into linear, hyperbolic, elliptic, and parabolic motion; and, in general, to describe any curve of the form $x\phi(x^2, y^2) + \psi(x^2, y^2) = 0$, where ϕ, ψ are homogeneous forms of functions of any degree respectively in x^2, y^2.

NOTE.—This is the admirably simple solution (discovered within the last few years by the French geometer Peaucellier) of a problem that had generally been supposed to be as ineffectuable as the quadrature of the circle, and the assumed impossibility of which many mathematicians of the first order had been long labouring to demonstrate; viz., the rigorously exact commutation of circular with rectilinear motion by means of a combination consisting exclusively of rods, joints, and points. If the second circle does not pass through the centre of the first, the fourth angle describes a circle instead of a straight line.

4320. (Proposed by J. J. SYLVESTER, F.R.S.)—If two points, taken equi-distant from one of the angles of a jointed equilateral rhomb on the adjacent sides, move in the same circle, and the opposite angle in a circle passing through the centre of the first, prove that the angle first named will move in a certain third circle.

I. *Solution by* G. S. CARR.

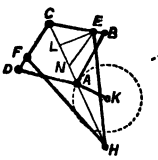

FIG. 1.

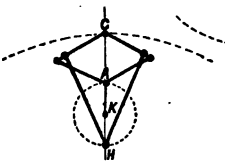

FIG. 2.
$q < n,\ a = b.$

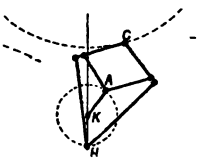

FIG. 3.
$q > n,\ a = b.$

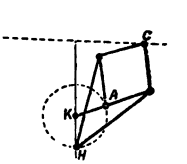

FIG. 4.
$n = q,\ a = b.$

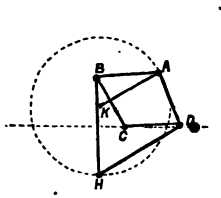

FIG. 5.
$n = q,\ a = b.$

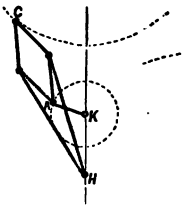

FIG. 6.
$n = q,\ a > b.$

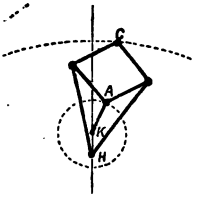

FIG. 7.
$n = q,\ a < b.$

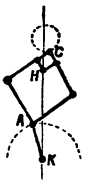

FIG. 8.
$p = q.$

As the discovery of Peaucellier's, and its extensions by Professor Sylvester, have excited considerable attention, I have made a brief investigation of all the cases in which the generated line is a straight line or circle, three rods being attached in each case to two fixed centres and also to a jointed rhombus.

Let H, K (Fig. 1) be the fixed centres; HE, HF equal rods attached to the rhombus at E, F, so that EC = CF. Let HK = a, AK = b, BC = n, HE = p, CE = q, AC = r, $\angle$ CHK = θ. Draw EL perpendicular to AC, and EN

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parallel to BA. Then we have

$$HC.HN = HL^2 - LC^2 = HE^2 - EC^2,$$

that is $$r\left\{HA + \frac{n-q}{n}(r - HA)\right\} = p^2 - q^2,$$

or $$(n-q)r^2 + qHAr = n(p^2-q^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the polar equation to the locus of the point C, in which equation HA is a function of θ. The following cases present themselves:

(i.) If $a=b$, we have $HA = 2a\cos\theta$, and the equation becomes

$$(n-q)r^2 + 2aqr\cos\theta - n(p^2-q^2) = 0.$$

Hence the point C moves in a circle whose centre lies in HK or HK produced at a distance from H $= \frac{a}{n-q}$ HK being the positive direction. We have figures (2) or (3) according as q is $<$ or $>$ n. The value of the radius is $$\frac{1}{n-q}\left\{a^2q^2 + n(n-q)(p^2-q^2)\right\}^{\frac{1}{2}},$$

and the locus becomes a point for any relation between the lengths of the rods which makes $a^2q^2 = n(q-n)(p^2-q^2)$, provided that q is not equal to n.

(ii.) If both $a=b$ and $n=q$, the equation becomes

$$2ar\cos\theta = p^2 - q^2,$$

and the locus of C is a straight line at right angles to HK and at a distance $\frac{p^2-q^2}{2a}$ from H. See Figures (4) and (5).

(iii.) If a be not equal to b, we have

$$HA = a\cos\theta + (b^2 - a^2\sin^2\theta)^{\frac{1}{2}}.$$

Thus equation (1) becomes

$$(n-q)r^2 + aqr\cos\theta - n(p^2-q^2) = -qr(b^2 - a^2\sin^2\theta)^{\frac{1}{2}}.$$

In this case, if $q=n$, we obtain

$$(a^2-b^2)r^2 - 2a(p^2-q^2)r\cos\theta + (p^2-q^2)^2 = 0,$$

the equation to a circle with its centre at a distance $\frac{a(p^2-q^2)}{a^2-b^2}$ from H on HK or KH produced, according as this distance is positive or negative. See Figures (6) and (7). The radius in this case is $= \frac{b(p^2-q^2)}{a^2-b^2}$, and the locus is a point if $p=q$.

(iv.) If $p=q$, equation (1) becomes

$$(n-q)^2r^2 + 2aq(n-q)r\cos\theta + q^2(a^2-b^2) = 0,$$

and the locus is a circle having the same centre as in case (i.), but a radius $= \frac{bq}{n-q}$. See Fig. (8). The construction in this case is identical in principle with the old-fashioned pentameter (so-called), and that instrument in fact contains the germ of Peaucellier's late invention.

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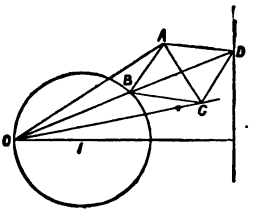
II. *Solution by* N'IMPORTE.



Two equal straight lines OA, OC turn about a fixed centre O. A jointed rhombus ABCD has two opposite angles A, C at the extremities of these lines. Then the two other angles B, D lie on a straight line through O, at right angles to AC; also OB . OD is constant, for if we draw a circle round C as centre, with radius CB, this will pass through D, and if OE be the tangent to this circle from O,

$$OB \cdot OD = OE^2 = OC^2 - CE^2 = OC^2 - CB^2,$$

which remains the same for all deformations. Hence, if we make B describe any curve whatever, D will describe the inverse curve with respect to O, and if B describe a circle passing through O, D will describe a straight line. If, then, we join B to a fixed point I, by a link equal to IO, D will describe a straight line.

4291. (Proposed by Professor TAIT.)—Upon what element of a planet's orbit, supposed nearly circular, will the attraction of masses external to the solar system produce the greatest effect.

Solution by G. M. MINCHIN, M.A.

Let S and M denote the positions of the sun and the disturbing body; SM = c, i' = inclination of SM to the plane of the fixed ecliptic, z = planet's distance from this plane, θ = planet's longitude measured from the projection of SM on the same plane; r = planet's radius vector.

Then, since the disturbing mass is supposed to be fixed, the potential, R, of the disturbing force is simply

$$M(r^2 - 2cr\cos i'\cos\theta + c^2 + z^2 - 2cz\sin i')^{-\frac{1}{2}},$$

M being the disturbing mass. Now c is, of course, extremely great; and we shall therefore find the potential of the disturbing force neglecting all powers of $\frac{r}{c}$ and $\frac{z}{c}$ beyond the second.

Let $r = a(1+u)$, $\theta = nt + \epsilon + v = \phi + v$, as in the Planetary Theory. Then, denoting the ratio $\frac{a}{c}$ by α, and putting $z = a\tan i \sin(\phi - \Omega)$, we may take

$$R_0 = (1 - 2\alpha\cos i'.\cos\phi + \alpha^2)^{-\frac{1}{2}} - \tfrac{1}{2}(1 - 2\alpha\cos i'.\cos\phi + \alpha^2)^{-\frac{3}{2}} \times [\tfrac{1}{4}\alpha^2\tan^2 i - \tfrac{1}{4}\alpha^2\tan^2 i\cos 2(\phi - \Omega) - 2\alpha\sin i'\tan i\sin(\phi - \Omega)];$$

and determine the complete value of R by the equation

$$R = R_0 + \frac{dR_0}{da}au + \tfrac{1}{1}\frac{dR_0}{d\phi}v + \tfrac{1}{2}\left(\frac{d^2R_0}{da^2}a^2u^2 + 2\frac{d^2R_0}{dad\phi}auv + \frac{d^2R_0}{d\phi^2}v^2\right).$$

Assuming that

$$(1 - 2\alpha\cos i'.\cos\phi + \alpha^2)^{-\frac{1}{2}} = \tfrac{1}{2}C_0 + C_1\cos\phi + C_2\cos 2\phi + \ldots\ldots$$

$$(1 - 2\alpha\cos i'.\cos\phi + \alpha^2)^{-\frac{3}{2}} = \tfrac{1}{2}D_0 + D_1\cos\phi + D_2\cos 2\phi + \ldots\ldots$$

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we shall find, for that part of R which gives rise to secular variations of the elements, the value

$$F = A_0 + A_1 e \cos \varpi + A_2 \tan i \sin \Omega + A_3 . e^2 + A_4 . \tan^2 i + A_5 . e^2 \cos 2\varpi + A_6 . \tan^2 i \cos 2\Omega,$$

in which $A_1 = -\left(C_1 + \frac{1}{2}\alpha \frac{dC_1}{d\alpha}\right)$, and all the other coefficients, are quantities of the second order in α, C_1 being of the first order, and therefore the most important term.

The determination of the elements to the second order of α is effected by the equations

$$\frac{de}{dt} = A_1 \sin \varpi + 2A_5 . e \sin 2\varpi, \quad e\frac{d\varpi}{dt} = A_1 \cos \varpi + 2A_3 . e + 2A_5 . e \cos 2\varpi,$$

$$\frac{d(\tan i)}{dt} = -A_2 \cos \Omega + 2A_6 . \tan i \sin 2\Omega,$$

$$\tan i \frac{d\Omega}{dt} = A_2 \sin \Omega - 2A_4 \tan i + 2A_6 . \tan i \cos 2\Omega.$$

Now since A_1 is much the most important coefficient in general, we see that the disturbance will be greatest in the excentricity and longitude of the perihelion.

If we neglect all the coefficients except A_1, we shall have

$$\frac{de}{dt} = A_1 \sin \varpi, \quad e\frac{d\varpi}{dt} = A_1 \cos \varpi.$$

These equations give $\tan \varpi = kt + \alpha$, where k and α are constants to be determined by observation, and $e = \frac{A_1}{k}\left\{1 + (kt+\alpha)^2\right\}^{\frac{1}{2}}$. Hence the excentricity is continually altered until it assumes such a value as will render our approximates invalid.

The magnitude of the alteration depends on the position of the disturbing body. It is easy to see that, neglecting α^2, the value of $C_1 + \frac{1}{2}\alpha \frac{dC_1}{d\alpha}$ is $\frac{3}{4}\alpha \cos i'$. Hence the greatest effect will be produced by a body in the plane of the ecliptic. For a body in the pole of the ecliptic $A_1 = 0$, and we must use the terms of the second order in α if we adhere to the supposition in the problem. It therefore appears that (the body being, of course, extremely distant) its disturbing effect is diminished according as it recedes from the plane of the ecliptic.

It is useless to trace the effect of the disturbance when the body is much removed from the plane of the ecliptic, for it is quite evident that the supposition of the sun's being uninfluenced by it would be inadmissible.

4209. (Proposed by A. MARTIN.)—One end of a heavy uniform chain is suspended from a peg in the side of a vertical wall. If the other end be moved on a horizontal line through the peg, find the locus of the middle point of the chain.

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Solution by the PROPOSER.

Put $2l$ = length of the chain, a = variable tension of the lowest point, and x and y the coordinates of a point in the required locus. Then, by the property of the catenary, we have

$$a = \frac{l^2-y^2}{2y}, \text{ and } AS = a \log\left(\frac{AV+SV}{AV-SV}\right), \text{ or } x = \left(\frac{l^2-y^2}{2y}\right) \log\left(\frac{l+y}{l-y}\right);$$

the equation to the locus required. (See Leybourn's *Mathematical Repository*, Vol. 3, pp. 138, 250, 251, Solution of Quests. 265, 327.)

4335. (Proposed by CHRISTINE LADD.)—From each vertex of a quadrilateral draw perpendiculars upon two sides and a diagonal, and show that the triangles formed by joining the feet of each set of perpendiculars are similar.

Solution by the PROPOSER.

Produce AA_c until it meets DC in E. The triangles DA_cE, AA_bE are similar; hence,

$$A_bE : A_cE = AE : DE.$$

Also AA_bDA_c is a quadrilateral inscribable in a circle, therefore

$$\angle DA_bA_c = A_cAD = \tfrac{1}{2}\pi - BDA,$$

and

$$\angle DA_cA_b = DAA_b = ADC - \tfrac{1}{2}\pi.$$

In the same way it may be shown that

$$\angle BA_cA_d = \tfrac{1}{2}\pi - ABC;$$

hence

$$\angle A_bA_cA_d = ABC + ADC;$$

similarly

$$\angle B_cB_dB_a = 2\pi - (BCD + BAD),$$

hence

$$\angle A_bA_cA_d = B_cB_dB_a.$$

The other angles of the triangles in question may also be easily proved equal to each other.

TABLES OF INVERSION AND RECIPROCATION OF A CURVE AND ITS PEDALS.
By J. L. MCKENZIE.

The accompanying tables may be of some interest as exhibiting concisely the results of alternate inversions and reciprocations of a curve U_0 and its successive pedals U_1, U_2 &c. The polar reciprocal of U_0 is repre-

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sented by V_0. Since the first pedal is the inverse of the polar reciprocal, reciprocation, followed by inversion, adds one to the suffix, and therefore shifts the U's one place to the left in the first table, and the V's one place to the right in the second. Again, since the first negative pedal is the polar reciprocal of the inverse, inversion, followed by reciprocation, subtracts one from the suffix, and therefore shifts the V's one place to the left in the first table, and the U's one place to the right in the second.

I.

1		U_0	U_1	U_2	U_3	U_{n-1}	U_n
2	Polar Recip. of (1)	V_0	V_{-1}	V_{-2}	V_{-3}	V_{-n+1}	V_{-n}
3	Inverse of (2)	U_1	U_2	U_3	U_4	U_n	U_{n+1}
4	Polar Recip. of (3)	V_{-1}	V_{-2}	V_{-3}	V_{-4}	V	V_{-n-1}
5	Inverse of (4)	U_2	U_3	U_4	U_5	U_{n+1}	U_{n+2}
	&c.						

II.

1		U_0	U_1	U_2	U_3	U_{n-1}	U_n
2	Inverse of (1)	V_1	V_0	V_{-1}	V_{-2}	V_{-n+2}	V_{-n+1}
3	Polar Recip. of (2)	U_{-1}	U_0	U_1	U_2	U_{n-2}	U_{n-1}
4	Inverse of (3)	V_2	V_1	V_0	V_{-1}	V_{-n+3}	V_{-n+2}
	&c.						

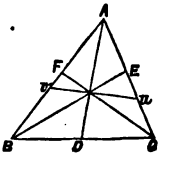
4350. (Proposed by Colonel CLARKE, C.B., F.R.S.)—Three points are taken at random in a given triangle: prove that the probability (p) that they enclose a given point whose triangular coordinates are (x, y, z), is

$$p = \mathrm{F}(x, y, z) + \mathrm{F}(y, z, x) + \mathrm{F}(z, x, y),$$

where $$\mathrm{F}(x, y, z) = 2xyz(1 - 6x^2) + (6y^2z^2 - 24y^3z^3)\log\left(1 + \frac{x}{yz}\right).$$

Solution by SEPTIMUS TEBAY, M.A.

Let ABC be the triangle, O the point within it, through which draw any straight line meeting AC, AB in u, v; and let AO, BO, CO meet the opposite sides in D, E, F.



Then, $$\Delta = \tfrac{1}{2}Au.Av \sin A + BOC + COu + BOv$$

$$= \frac{Au}{AC} \cdot \frac{Av}{AB}\Delta + BOC + COu + BOv;$$

therefore $$\frac{u}{y} \cdot \frac{v}{z} + x + y - u + z - v = 1;$$

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putting $\frac{\text{AO}u}{\Delta} = u$, &c. Since $x+y+z=1$, this expression may be written

$$(u-yz)(v-yz) = y^2z^2.$$

Let $v=z$; then $u = \frac{yz}{1-y} = \frac{\text{AOE}}{\Delta}$; which will be useful in assigning the limits of integration.

For points in COE, COD, BOF, the chance is

$$\frac{6xy}{1-x}\int_{\frac{yz}{1-y}}^{y}\left(v-\frac{zy}{1-z}\right)du = \frac{6xy^2z^2}{1-x}\log\left(1+\frac{x}{yz}\right) - \frac{6x^2y^2z^2}{(1-x)(1-y)(1-z)}y.$$

For COE, BOD, BOF,

$$\frac{6xz}{1-x}\int_{\frac{yz}{1-y}}^{y}\left(v-\frac{zy}{1-z}\right)du = \frac{6xy^2z^2}{1-x}\log\left(1+\frac{x}{yz}\right) - \frac{6x^2y^2z^2}{(1-x)(1-y)(1-z)}z.$$

For COE, BOF, AOF,

$$\frac{6zy}{1-z}\int_{\frac{yz}{1-y}}^{y}(z-v)\,du = \frac{6xy^2z^2}{1-z} - \frac{6y^3z^3}{1-z}\log\left(1+\frac{x}{yz}\right).$$

For COE, BOF, AOE,

$$\frac{6yz}{1-y}\int_{\frac{yz}{1-y}}^{y}(z-v)\,du = \frac{6xy^2z^2}{1-y} - \frac{6y^3z^3}{1-y}\log\left(1+\frac{x}{yz}\right).$$

For COE, AOv, BOv,

$$6\int_{\frac{yz}{1-y}}^{y}(z-v)\left(v-\frac{zy}{1-z}\right)du = \frac{6y^2z^3}{1-z}(x+2yz)\log\left(1+\frac{x}{yz}\right) - \frac{12xy^2z^3}{1-z}.$$

Similarly for BOF, AOu, COu, the chance is

$$\frac{6y^3z^2}{1-y}(x+2yz)\log\left(1+\frac{x}{yz}\right) - \frac{12xy^3z^2}{1-y}.$$

The sum of these parts is

$$6xy^2z^2\left(4-\frac{1}{1-y}-\frac{1}{1-z}\right) + (6y^2z^2-24y^3z^3)\log\left(1+\frac{x}{yz}\right) - \frac{6x^2y^2z^2}{(1-x)(1-y)(1-z)}(y+z).$$

There are similar expressions in zx, xy; namely,

$$6yz^2x^2\left(4-\frac{1}{1-z}-\frac{1}{1-x}\right) + (6z^2x^2-24z^3x^3)\log\left(1+\frac{y}{zx}\right) - \frac{6x^2y^2z^2}{(1-x)(1-y)(1-z)}(z+x),$$

$$6zx^2y^2\left(4-\frac{1}{1-x}-\frac{1}{1-y}\right) + (6x^2y^2-24x^3y^3)\log\left(1+\frac{z}{xy}\right) - \frac{6x^2y^2z^2}{(1-x)(1-y)(1-z)}(x+y).$$

The sum of the algebraical portions of these is

$$6xyz(1-2x^2-2y^2-2z^2) - \frac{12x^2y^2z^2}{(1-x)(1-y)(1-z)}.$$

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But for points in AOF, BOD, COE, the chance is

$$6\,\frac{zy}{1-z}\cdot\frac{xz}{1-x}\cdot\frac{yx}{1-y}=\frac{6x^2y^2z^2}{(1-x)(1-y)(1-z)};$$

and for points in AOE, BOF, COD, the chance is

$$6\,\frac{yz}{1-y}\cdot\frac{zx}{1-z}\cdot\frac{xy}{1-x}=\frac{6x^2y^2z^2}{(1-x)(1-y)(1-z)}.$$

Hence the whole algebraical portion is

$$6xyz\,(1-2x^2-2y^2-2z^2);$$

and therefore the probability is as enunciated.

4358. (Proposed by J. L. McKenzie.)—1. A given ellipse U is one of a system of concentric, similar, and similarly situated ellipses. A line is drawn touching any other ellipse V of the system; and the perpendicular distance of the tangent from the centre is a mean proportional between the semi-axis major of V and the semi-axis minor of U. Show that the envelope of the tangent is the first negative pedal of U, but rotated through a right angle about its centre.

2. If the duplicate ratio of the perpendicular on the tangent to the linear unit, be equal to the ratio of linear similarity of V and U, show that the envelope of the tangent is the first negative pedal of the polar reciprocal of U.

I. *Solution by the* Editor.

In Dr. Booth's *New Geometrical Methods*, p. 159, it is shown that the tangential equation of the first negative pedal may be found by substituting $\left(\frac{\xi}{\xi^2+v^2}\right)$ and $\left(\frac{v}{\xi^2+v^2}\right)$ for x and y in the projective equation of the curve. Thus let the projective equation of an ellipse be

$$a^2y^2+b^2x^2=a^2b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then the tangential equation of its first negative pedal will be

$$a^2v^2+b^2\xi^2=a^2b^2(\xi^2+v^2)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If a be taken as the radius of the polarizing circle, the reciprocal polar of (1) is

$$b^2y^2+a^2x^2=a^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3);$$

and the tangential equation of its first negative pedal will be

$$b^2v^2+a^2\xi^2=a^4(\xi^2+v^2)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Now let

$$U\equiv a^2\xi^2+b^2v^2-1=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

be the tangential equation of the given ellipse, and let

$$V\equiv n^2a^2\xi^2+n^2b^2v^2-1=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

be the tangential equation of one of the similar ellipses. Then, by the condition of the question, we have

$$nab\,(\xi^2+v^2)=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Eliminating n from (6) and (7) we get

$$a^2\xi^2+b^2v^2=a^2b^2(\xi^2+v^2)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

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which would coincide with (2) were this latter turned through a right angle.

Again, let $n = a^2(\xi^2 + v^2)$ ..... (9).

Substituting this value of n in (6) we get

$$a^2\xi^2 + b^2v^2 = a^4(\xi^2 + v^2)^2$$

which is identical with (4).

II. *Solution by the* PROPOSER.

1. Let a and b be the semi-axes of U, ma and ma of V. Then the tangential equation of V is $a^2\xi^2 + b^2v^2 = \frac{1}{m^2}$ ..... (1).

The distance of the tangent from the origin is $(mab)^{\frac{1}{2}}$; therefore the tangent touches the circle whose tangential equation is

$$\xi^2 + v^2 = \frac{1}{mab} \quad \ldots\ldots (2).$$

Eliminating $\frac{1}{m}$ between (1) and (2), we obtain

$$a^2\xi^2 + b^2v^2 = a^2b^2(\xi^2 + v^2)^2 \quad \ldots\ldots (3).$$

This curve is the polar reciprocal of

$$a^2x^2 + b^2y^2 = a^2b^2(x^2 + y^2)^2 \ldots\ldots (4).$$

Now (4) is the inverse curve of the ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \ldots\ldots (5),$$

and since the polar reciprocal of the inverse of a curve is the first negative pedal, (3) represents the first negative of (5), that is of U turned about its centre through a right angle.

2. In this case the distance of the perpendicular from the centre is $m^{\frac{1}{2}}$; therefore the circle touched by the tangent is

$$\xi^2 + v^2 = \frac{1}{m} \quad \ldots\ldots (6);$$

and eliminating $\frac{1}{m}$ between this and (1), we have

$$\frac{\xi^2}{a^2} + \frac{v^2}{b^2} = \frac{1}{a^2b^2}(\xi^2 + v^2)^2 \quad \ldots\ldots (7),$$

the first negative of $a^2x^2 + b^2y^2 = 1$ ..... (8),

that is, of the polar reciprocal of U.

4321. (Proposed by the EDITOR.)—1. A constant angle α revolves round its vertex, which is fixed at a focus of a conic whose excentricity is e, and semi-parameter p; and a tangent is drawn at the point where the conic is cut by one of the sides of the angle; show that the locus of

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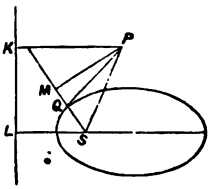
the point in which this tangent meets the other side of the angle is a conic whose excentricity is $e \sec \alpha$, and semi-parameter $p \sec \alpha$.

2. The base of a triangle, and the sum or the difference of the sides, are given; find the locus of the point in which the bisector of the exterior or the interior vertical angle meets a straight line making a given angle with one of the sides, and drawn through one end of the base.

I. *Solution by* Professor WOLSTENHOLME; CHRISTINE LADD; *and others.*

1. Let S be the focus of the conic, KL the corresponding directrix, QSP ($= \alpha$) the given revolving angle, QP the tangent at Q, and P the point whose locus is required. Draw PM, PK perpendicular to SQ, KL respectively. Then it is proved in all treatises on Geometrical Conics, that $SM = e \cdot PK$; but $SM = SP \cdot \cos \alpha$, therefore $SP = e \sec \alpha \cdot PK$; hence the locus of P is a conic with the same focus and directrix as the original one, and with excentricity $e \sec \alpha$, whence also its semi-parameter is $p \sec \alpha$.



2. This is only another way of stating what is proved in the first part of the question.

II. *Solution by the* Rev. Dr. BOOTH, F.R.S.

Let $$b^2(\xi^2 + v^2) + 2ae\xi - 1 = 0 \equiv \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

be the tangential equation of an ellipse, the focus being the origin. Let τ be the tangent of the revolving angle, then the coordinates (x, y) of the point in which the side of the revolving angle meets the tangent to the curve drawn at the point in which the other side of the revolving angle meets the curve, are given by the general equations*

$$x = \frac{1}{\Delta}\left(\frac{d\phi}{d\xi} - \tau\frac{d\phi}{dv}\right), \quad y = \frac{1}{\Delta}\left(\frac{d\phi}{dv} + \tau\frac{d\phi}{d\xi}\right) \quad \ldots\ldots\ldots (b),$$

where $$\Delta = \frac{d\phi}{d\xi}\xi + \frac{d\phi}{dv}v + \tau\left[\frac{d\phi}{d\xi}v - \frac{d\phi}{dv}\xi\right].$$

Now $\frac{d\phi}{d\xi} = 2(b^2\xi + ae)$, and $\frac{d\phi}{dv} = 2b^2v$, consequently

$$\frac{d\phi}{d\xi}\xi + \frac{d\phi}{dv}v = 2(1 - ae\xi), \text{ and } \frac{d\phi}{d\xi}v - \frac{d\phi}{dv}\xi = 2aev \quad \ldots\ldots\ldots (c).$$

Substituting these values in the general form, we have

$$x = \frac{b^2\xi + ae - \tau b^2 v}{1 - ae\xi + \tau aev}, \quad y = \frac{b^2 v + \tau b^2\xi + \tau ae}{1 - ae\xi + \tau aev} \quad \ldots\ldots\ldots\ldots (d).$$

and these, by multiplication, may be reduced to

$$x - ae\xi x + \tau aevx = b^2\xi + ae - \tau b^2 v, \quad y - ae\xi y + \tau aevy = b^2 v + \tau b^2\xi + ae\tau \ldots (e).$$

The dual equation $x\xi + yv = 1$ gives, writing χ for $\frac{v}{\xi}$,

$$x + \chi y = \frac{1}{\xi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (f).$$

* See BOOTH'S *New Geometrical Methods*, p. 168.

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Dividing the preceding equations by ξ, writing for $\frac{v}{\xi}$ its value χ and eliminating $\frac{1}{\xi}$ by the preceding equation, we get

$$\left.\begin{aligned}&[\tau aex+\tau b^2-aey+xy]\,\chi = b^2+2aex-x^2\\ \text{and}\quad&[y^2-b^2]\,\chi = \tau b^2+\tau aex+aey-yx\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (g).$$

Eliminating χ by simple division, we finally obtain, after some cancelling of equal terms,

$$\frac{a^2y^2}{b(1+\tau^2)} + \frac{a^2[1-e^2(1+\tau^2)]\,x^2}{b^4(1+\tau^2)} - \frac{2aex}{b^2} = 1 \quad \ldots\ldots\ldots\ldots (h).$$

But the form of the projective equation of an ellipse, the focus being the origin, and A and B being the semi-axes, is

$$\frac{A^2y^2}{B^4} + \frac{x^2}{B^2} - \frac{2Aex}{B^2} = 1 \quad \ldots\ldots\ldots\ldots (i).$$

Comparing this form with the preceding equation, and putting $\sec^2\alpha$ for $1+\tau^2$, we shall have

$$B^2 = \frac{b^4\sec^2\alpha}{a^2(1-e^2\sec^2\alpha)}, \quad \frac{B^4}{A^2} = \frac{b^4}{a^2}\sec^2\alpha \quad \ldots\ldots\ldots\ldots (j);$$

therefore the new parameter $= \frac{B^2}{A} = \frac{b^2}{a}\sec\alpha$. We have also

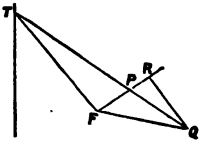
$$A = \frac{B^2a}{b^2\sec\alpha} = \frac{b^2\sec\alpha}{a(1-e^2\sec^2\alpha)},$$

and as $e_1^2 = 1-\frac{B^2}{A^2} = 1-(1-e^2\sec^2\alpha)$, therefore $e_1 = e\sec\alpha$.

However the angle α may vary, the coefficient of x continues invariable, inasmuch as we have $\frac{ae}{b^2} = \frac{Ae_1}{B^2}$.

III. *Solution by* JOHN C. MALET, M.A.

1. In the figure let P be a point on the conic, F the focus. Draw FT perpendicular to FP, meeting the corresponding directrix in T, then PT is the tangent to the conic at P. Take another point Q on the tangent, and on FP let fall a perpendicular QR; let the angle $PFQ = \alpha$. Since the triangles FPT and QPR are similar, we have evidently



$$\frac{FP}{PT} = \frac{FR}{QT} = \frac{FQ}{QT}\cos\alpha, \text{ or } \frac{FQ}{QT} = \frac{FP}{PT}\sec\alpha.$$

From this we see the locus of Q is a conic with the same focus and directrix as the given conic, and the excentricity $= e\sec\alpha$.

2. This is the application of (1) to the cases of the ellipse and hyperbola. In each case the locus is a conic.

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IV. *Solution by* J. L. McKenzie.

1. This theorem is the reciprocal of the following:—"If a line T touch a circle of radius r, and through the point of contact another line L be drawn making with T a constant angle α, the envelope of L is a concentric circle of radius $r \cos \alpha$." If the reciprocal of the envelope of T be the given conic, the reciprocal of the envelope of L will be the required locus, a conic having the same directrix. The ratio of the excentricities of the two conics is $\frac{1}{r} : \frac{1}{r \cos \alpha} = 1 : \sec \alpha$. Since they have the same focus and directrix, their parameters are proportional to their excentricities.

2. If the given base $= 2c$ and the sum of the sides $= 2a$, the locus of the vertex of the triangle is a conic of excentricity $= \frac{c}{a}$ and semi-parameter $= \frac{a^2 - c^2}{a}$; and the bisector of the exterior or interior vertical angle is the tangent; hence, by the former part of the question, the required locus is a conic with one focus at that extremity of the base about which the constant angle revolves, and with excentricity $= \frac{c}{a \cos \alpha}$ and semi-parameter $\frac{a^2 - c^2}{a \cos \alpha}$. The conic is therefore an ellipse, parabola, or hyperbola according as $\cos \alpha$ is greater than, equal to, or less than $\frac{c}{a}$, that is, than the cosine of the angle at the base of the moveable triangle when it becomes isosceles.

If $2a$ is the *difference* of the sides, a is always less than c, and the required locus is always an hyperbola.

V. *Solution by the* Proposer.

1. Let ϕ be the inclination of that side of the angle at whose intersection with the conic the tangent is drawn, (ρ, θ) the polar coordinates of any point in the tangent, and (r, ω) those of the required locus. Then, p being the semi-parameter of the conic, the polar equation of the tangent is

$$\rho \left\{ \cos (\theta - \phi) - e \cos \theta \right\} = p.$$

Now when $\rho = r$, then $\theta = \omega = \phi \pm \alpha$; hence the preceding becomes

$$r (\cos \alpha - e \cos \omega) = p;$$

or, putting $p \sec \alpha = p'$, and $e \sec \alpha = e'$, the equation of the locus becomes $r (1 - e' \cos \omega) = p'$; whence we see that the locus is a conic whose parameter is $2p'$, and excentricity e', and that it is an ellipse, a parabola, or an hyperbola, according as $\cos \alpha > = < e$.

2. The first part of the Question includes the following one:—The base of a triangle, and (1) the sum, (2) the difference of the sides, are given: to find the locus of the point in which the bisector of (1) the exterior, (2) the interior vertical angle, meets a straight line making a given angle with one of the sides, and drawn through one end of the base. When the given angle is a right angle (or $\alpha = \frac{1}{2}\pi$, $\cos \alpha = 0$) the equation of the locus becomes $p + re \cos \omega = 0$; and the locus is then a straight line perpendicular to the base.

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4152. (Proposed by PAUL SERRET.)—Une ellipse et l'un de ses cercles directeurs étant tracés, il existe une infinité de triangles simultanément sincrits au cercle et circonscrits à l'ellipse; le point de rencontre des hauteurs est le même pour tous ces triangles.

I. *Solution by* S. FORDE.

Writing $S_1 \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, $S_2 \equiv x^2 + y^2 - r^2$, we find for the invariants

$$\Delta_1 = -\frac{1}{a^2b^2}, \quad {}_1\Theta_2 = -\frac{1}{a^2b^2}(a^2+b^2+r^2), \quad -{}_2\Theta_1 = 1 + r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right);$$

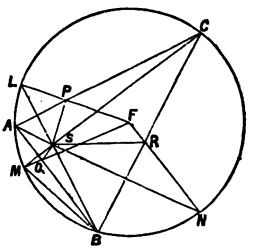
and the condition ${}_1\Theta_2{}^2 = 4_1\Delta_2\Theta_1$ becomes

$$(a^2+b^2+r^2)^2 - 4\left\{a^2b^2 + r^2(a^2+b^2)\right\}, \text{ or } r^4 - 2r^2(a^2+b^2) + (a^2-b^2)^2 = 0;$$

whence $$r^2 = (a \pm b)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If (α, β) denote the centre of perpendiculars of one of the triangles, and $S_3 \equiv (x-\alpha)^2 + (y-\beta)^2 - r'^2$ the circle with regard to which the triangle is self conjugate, we have ${}_1\Theta_3 = 0$, and ${}_3\Theta_2 = 0$, since there exist an infinite number of triangles which are simultaneously self-conjugate with regard to S_3, inscribed in S_2, and circumscribed about S_1. We find the invariants to be ${}_1\Theta_3 = \alpha^2 + \beta^2 - a^2 - b^2 - r'^2 = 0$, ${}_3\Theta_2 = \alpha^2 + \beta^2 - r^2 - 2r'^2 = 0$, whence eliminating r'^2, the locus of the centre of perpendiculars is found to be the circles $\alpha^2 + \beta^2 = 2(a^2+b^2) - r^2 = (a \mp b)^2$ by (1).

II. *Geometrical Solution by* J. M. WILSON, M.A.



Let F be the centre of a director-circle; AP, AQ tangents from A to the ellipse at P, Q; and let FP, FQ meet the circle in L, M. Let S be the other focus. Then AS=AL=AM. Also LS is perpendicular to AC, and SM to AB. Let LS and AB meet in B, and join MB. Then, since AB bisects SM at right angles, the angle SBA = MBA; and the arcs AL, AM are equal; therefore B lies on the director-circle. Similarly, if MS meets AP produced in C, C is on the director-circle. Join AS, and produce it to meet the circle in N. Join FN, cutting BC in R, and join SR, CN. Then NCB = NAB = MCB. Therefore BC bisects SN at right angles; therefore

$$FR + RS = FN = FL = FP + SP,$$

and FR, SR make equal angles with BC; that is, an ellipse could be described with foci F, S to touch the sides of ABC in P, Q, R; which proves the theorem.

4272. (Proposed by C. W. MERRIFIELD, F.R.S.)—From the corners of a triangle ABC lines are drawn through a point O, meeting the opposite

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sides in DEF. Through A a line B_1C_1 is drawn parallel and equal to BC; and OB_1, OC_1 are drawn to meet BC in B_2C_2. If C_3A_2, A_3B_3 are analogous segments on CA, AB respectively; prove (1) that

$$AF \cdot BD \cdot CE = BF \cdot CD \cdot AE = B_2C_2 \; C_3A_2 \cdot A_3B_3,$$

and (2) find a corresponding theorem in regard to lines drawn through the angles parallel to any transversal DEF.

I. *Solution by* J. J. WALKER, M.A.

1. Evidently $B_2C_2 = \frac{OD \cdot BC}{OA}$; but, considering the three sides of the triangle ABD as cut by the transversal COF, $\frac{OD \cdot BC}{OA} = \frac{CD \cdot BF}{AF}$. Similarly, if C_3A_2 be the analogous segment on CA, and A_3B_3 that on AB,

$$C_3A_2 = \frac{AE \cdot CD}{BD}, \text{ and } A_3B_3 = \frac{AF \cdot BD}{CD}.$$

Hence $B_2C_2 \cdot C_3A_2 \cdot A_3B_3 = BF \cdot AE \cdot CD = AF \cdot BD \cdot CE.$

2. Let AG, BH, CI, all parallel to a transversal DEF (which cuts BC in D, CA in E, and AB in F), meet BC in G, CA in H, and AB in I respectively. Then $\frac{BI}{BF} = \frac{BC}{BD}$, $\frac{AH}{AE} = \frac{AB}{AF}$ and $\frac{CG}{CD} = \frac{CA}{CE}$,

consequently $$\frac{AH \cdot BI \cdot CG}{AE \cdot BF \cdot CD} = \frac{AB \cdot BC \cdot CA}{AF \cdot BD \cdot CE};$$

whence $$AH \cdot BI \cdot CG = AB \cdot BC \cdot CA.$$

Similarly it may be shown that

$$AI \cdot BG \cdot CH = AB \cdot BC \cdot CA.$$

Hence $$AH \cdot BI \cdot CG = AI \cdot BG \cdot CH.$$

II. *Solution by the* PROPOSER.

In a triangle ABC, AO, BO, CO produced meet the sides in D, E, F, and the sides are divided in the ratios $l : m : n$. Now take the compound ratio

$$\frac{l}{m+n} \cdot \frac{m}{n+l} \cdot \frac{n}{l+m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and permute the denominator; then we get

$$\frac{OD \cdot OE \cdot OF}{OA \cdot OB \cdot OC} = \frac{AF \cdot BD \cdot CE}{abc} = \frac{BF \cdot CD \cdot AE}{abc} \quad \ldots\ldots\ldots (2);$$

and we want to find lines such that

$$OD : OA = a : a, \text{ \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

For this purpose, through A draw a line B′C′ parallel and equal to BC. Join OB′, OC′, meeting BC in B″, C″, and so symmetrically. Then

$$B''C'' \cdot OA = OD \cdot BC,$$

and $$AF \cdot BD \cdot CE = BF \cdot CD \cdot AE = B''C'' \cdot C''A'' \cdot A''B''.$$

The corresponding transversal theorem is this:—Through the angles of the triangle ABC draw lines parallel to the transversal DEF, meeting the opposite sides in α, β, γ; also, let AL, BM, CN be perpendiculars on the

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transversal; then, by permuting the denominator of $\frac{AL}{\sin D} \cdot \frac{BM}{\sin E} \cdot \frac{CN}{\sin F}$,

we get $$AF \cdot BD \cdot CE = AE \cdot BF \cdot CD = \alpha D \cdot \beta E \cdot \gamma F.$$

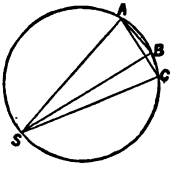
It will be seen that α, β, γ depend on the direction, and not on the position, of the transversal.

4268. (Proposed by Professor Wolstenholme, M.A.)—If A, B, C be three points on a parabola; A′B′C′ the triangle formed by the tangents at A, B, C; and a, b, c the centres of the circles BCA′, CAB′, ABC′; prove that the circle abc will pass through the focus.

Solution by F. D. Thomson, M.A.

1. If S, A, B, C be four points on a circle, then

$$SA \cdot BC + SC \cdot BA = SB \cdot AC,$$

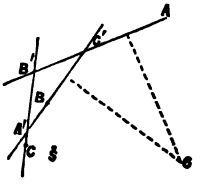


or, since

$$BC : CA : AB = \sin BAC : \sin CBA : \sin ACB$$
$$= \sin BSC : \sin CSA : \sin ASB,$$
$$SA \sin BSC + SC \sin ASB = SB \sin CSA \quad \ldots(1);$$

and conversely, if the relation (1) hold between the vectors from S and the angles between them, the four points S, A, B, C are on a circle.

2. Let A′B′C′ be the triangle formed by the tangents at A, B, C to the parabola with focus S. Then, in the triangle A′B′C′, $\angle C' = \frac{1}{2} \angle BSA$. But $\angle C' = \frac{1}{2} \angle$ at centre of circle circumscribing AC′B,



therefore $$\angle BSA = \angle \text{at centre},$$

therefore, if c be the centre of this circle, a circle will pass round ABSc;

therefore $$Sc \cdot AB + SB \cdot Ac = SA \cdot Bc.$$

But $$cA = cB = \frac{AB}{2 \sin C'},$$

therefore $$2Sc \sin C' = SA - SB.$$

Similarly, if a and b be the centres of the other circumscribing circles,

$$2Sa \sin A' = SB - SC \quad \text{and} \quad 2Sb \sin B' = SA - SC,$$

therefore $$Sc \sin C' + Sa \sin A' = Sb \sin B' \quad \ldots\ldots\ldots\ldots (2).$$

But since SC′ bisects the angle BSA, it passes through the middle point F of the arc AFB of the circle ABSc and cF is a diameter, therefore Sc is perpendicular to SC′.

Therefore Sa, Sb, Sc make the same angles with each other as do SA′, SB′, SC′. But a circle will pass round SA′B′C′, therefore these angles are equal to the angles in the triangle A′B′C′; therefore, substituting in (2), we have $$Sa \sin bSc + Sc \sin aSb = Sb \sin cSA.$$

Hence, comparing with (1), we see that a, b, c, S lie on a circle.

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4326. (Proposed by Rev. Dr. Booth, F.R.S.)—Let a, b, c, d be the sides, and S the area of a quadrilateral circumscribed about one circle, and inscribed in another of radius R; prove that

$$S^2 = abcd, \text{ and } 16\,R^2 = a^2+b^2+c^2+d^2+S^2\,(a^{-2}+b^{-2}+c^{-2}+d^{-2}).$$

Solution by M. Collins, LL.D.; A. Martin; *and others.*

Let a, b, c, d be the sides, and A, B, C, D the angles of *any* quadrilateral taken in order, and $2s = a+b+c+d$; then it is easily proved that

$$(\text{Area})^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\tfrac{1}{2}(A+C);$$

therefore, when a, b, c, d are given, the area is a maximum when $A+C=180°$, and then we have $(\text{maximum area})^2 = (s-a)(s-b)(s-c)(s-d)$,

as may in fact be easily proved geometrically. Now, as by hypothesis a circle can be inscribed in the quadrilateral, we have $a+c = b+d = s$, and therefore $s-a = c$, $s-b = d$, &c.; hence $S^2 = abcd$. Again, if x be the diagonal which forms a triangle with a and b, and y the other diagonal, we have

$$xy = ac+bd \text{ and } \frac{x}{y} = \frac{ad+bc}{ab+cd};$$

and the product of these two equations gives

$$x^2\,(ab+cd)^2 = (ab+cd)(ac+bd)(ad+bc) = \Sigma a^3bcd + \Sigma a^2b^2c^2$$
$$= S^2\,(a^2+b^2+c^2+d^2) + S^4\,(a^{-2}+b^{-2}+c^{-2}+d^{-2}).$$

Now the radius of the circle about a triangle being $= \dfrac{\text{product of 3 sides}}{4\Delta}$

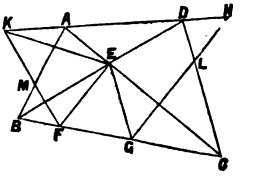
we get $4R = \dfrac{abx}{\Delta abx} = \dfrac{cdx}{\Delta cdx} = \dfrac{(ab+cd)\,x}{S}$, and therefore

$$(4R)^2 = x^2\,(ab+cd)^2 - S^2 = a^2+b^2+c^2+d^2+S^2\,(a^{-2}+b^{-2}+c^{-2}+d^{-2}).$$

4282. (Proposed by F. H. Dennis, M.A.)—ABCD is a quadrilateral inscribed in a circle. A circle round DCE (E being the intersection of the diagonals) meets BC, AD in G and H respectively, and GH intersects CD in L. A circle round ABE meets BC, AD in F and K respectively, and FK intersects AB in M. Prove that LM is parallel to L′M′, where L′, M′ are points similarly determind by circles round AED, BEC.

I. *Solution by the* Proposer.

The angle EKF = EBF = EAD = EFK, therefore EK = EF; also the angle EGF = EDC = EAB = EFG, therefore EG = EF = EK = EH; hence E is the centre of a circle passing through F, G, H, K.



Now L is the intersection of the common chords of the circles (DECH, ABCD) and (DECH, KFGH); hence the chord of intersection of (ABCD, KFGH) passes through

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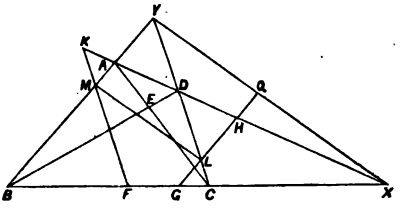
L. Similarly it passes through M. Therefore LM is the chord of intersection of ABCD, KFGH. Similarly L′M′ is the chord of intersection of ABCD and the concentric circle K′F′G′H′; therefore LM is parallel to L′M′.

II. *Solution by* F. D. Thomson, M.A.

Let X and Y be the intersections of the opposite sides AD, BC and BA, CD; join XY, and produce GL to meet XY in Q.



Then, since angles in the same segments are equal,

$\angle LGC = HDC = ABC$;

therefore HG is parallel to AB; similarly FG is parallel to CD.

Hence by similar triangles

$$\frac{LH}{AY} = \frac{HD}{DA} = \frac{HA}{DA} - 1, \quad \text{and} \quad \frac{HQ}{AY} = \frac{XH}{XA} = 1 - \frac{HA}{XA};$$

therefore adding,

$$\frac{LQ}{AY} = \frac{XD \cdot HA}{XA \cdot DA}.$$

But, taking the intercepts which BCX makes on the sides of the triangle AED, we have

$$DX \cdot EB \cdot AC = DB \cdot EC \cdot AX;$$

therefore

$$\frac{LQ}{AY} = \frac{DB \cdot EC \cdot HA}{EB \cdot AC \cdot DA} = \frac{DB \cdot EC \cdot HA \cdot DA}{EB \cdot AC \cdot DA^2}$$

$$= \frac{DB \cdot EC \cdot AC \cdot AE}{EB \cdot AC \cdot DA^2} = \frac{DB \cdot BE \cdot ED}{EB \cdot DA^2} = \frac{DB \cdot ED}{DA^2}$$

$$= \frac{DA \cdot DK}{DA^2} = \frac{DK}{DA} = \frac{YM}{YA} \text{ (Euc. iii. 36, and similar triangles).}$$

Therefore LQ = YM. But LQ is parallel to YM; therefore LM is parallel to XY. Similarly, it may be shown that L′M′ is parallel to XY. Therefore, &c.

4339. (Proposed by Professor Townsend, F.R.S.)—Show that the two principal axes at the centre of gravity, in the plane of a uniform triangular lamina, are the two rectangular conjugates of the involution determined by the three pairs of lines through the point which bisect internally and externally the three sides of the triangle.

I. *Solution by* J. J. Walker, M.A.

Taking, as axes of x and y, straight lines through the centre of gravity parallel and perpendicular respectively to the base BC of the triangle, and $x^2 + \mu xy - y^2 = 0$ being the equation to a pair of straight lines through the

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origin at right angles one to the other, the perpendiculars on which, from any element dm of the mass of the lamina, are q, q'; the condition

$\Sigma qq'dm = 0$ gives $\int_{y''}^{y'} \int_{x''}^{x'} (x^2 + \mu xy - y^2)\, dx\, dy = 0,$

the limits being defined by

$$y' = \tfrac{2}{3}p, \quad y'' = -\tfrac{1}{3}p, \quad x' = \tfrac{1}{3}a - \cot \mathrm{B}y, \quad x'' = \tfrac{1}{3}a + \cot \mathrm{C}y,$$

where $a = \mathrm{BC}$, and p is the perpendicular from A on BC.

The value of the double integral above is found, after obvious reductions, to be $\frac{\mu}{72} ap^3 (\cot \mathrm{C} - \cot \mathrm{B}) - \frac{ap}{36}\left\{a^2 - (\cot \mathrm{B} \cot \mathrm{C} + 1)\, p^2\right\}$,

which, equated with zero, gives

$$\mu = \frac{2\left\{(\cot \mathrm{C} + \cot \mathrm{B})^2 - \cot \mathrm{C} \cot \mathrm{B} - 1\right\}}{\cot \mathrm{C} - \cot \mathrm{B}};$$

but this is easily found to be the condition that $x^2 + \mu xy - y^2$ should be the two rectangular conjugates of the involution determined by the pairs of bisectors of sides of the triangle through the origin:

viz., $xy - \cot \alpha y^2 = 0$, $x^2 + (\cot \beta - \cot \mathrm{C})\, xy - \cot \beta \cot \mathrm{C}\, y^2 = 0$,

and $x^2 + (\cot \mathrm{B} - \cot \gamma)\, xy - \cot \gamma \cot \mathrm{B} y^2 = 0$;

α, β, γ being the angles which the internal bisectors, passing through A, B, C, respectively make with BC.

II. *Solution by the* Proposer.

For the three pairs of lines in question being evidently three pairs of axes for which the three products of inertia of the lamina vanish, and all such pairs of axes at any point being conjugate diameters of the ellipse of inertia of the lamina at the point, therefore &c.

For the same reason, the two principal axes at the centre of gravity of a uniform rectangular lamina, or more generally of one in the form of any parallelogram, are the two rectangular conjugates of the involution determined by the two diagonals and by the two concurrent parallels to the two pairs of opposite sides of the figure.

On the phrase "a Random Line." *By* G. S. Carr.

I have recently read some observations by Professor Crofton on "Local Probability," which are given in Vol. VII. of the *Reprint*, pp. 84—86. Had my attention been directed to these and kindred remarks before, and had I been aware that a precedent had been established in this Journal for solving such probability questions as 3693 in the way adopted by the Proposer, I should have approached the subject somewhat differently, if indeed I had thought it expedient to attempt to revive an old discussion.

If, after a careful consideration of what has been written on the subject of "Random Lines," I venture to call in question the fitness of the defini-

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tion established by writers in these pages, I do so with the utmost deference to the views of older contributors than myself.

For the sake of clearness in the argument which I am about to advance, I will first state the usual law of probability in the case of a single event.

If an event can happen in m ways only, all of which are equally probable, and if the same event can happen in n, out of those m ways, under certain restrictive conditions, then the probability of the restricted event happening is $\frac{n}{m}$. But if the m ways are not equally probable, $\frac{n}{m}$ will not generally represent the probability in question. The m way may then be sub-divided into groups of ways, each of which groups must fulfil the condition that all the ways it includes are equally probable. The probability of any particular group being selected, multiplied by a fraction similar in kind to $\frac{n}{m}$ will express the probability of the restricted event happening in that particular group; and the sum of the probabilities calculated in this way for each group will be the true probability of the restricted event happening at all.

Question 3693, by Mr. Watson, is expressed as follows, excepting that I have substituted the word "area" for "triangle," for the sake of generalizing the problem as Professor Crofton has done in Question 4119:—"A line is drawn at random, so as to cut a given area, and two points are taken at random within the area. Find the probability that the two points lie on opposite sides of the line."

In Vol. VII. of the *Reprint*, p. 84, Professor Crofton gives a general definition of a line drawn at random in an unlimited plane, and thence in a foot-note defines "a line drawn at random in a given area" as "a line taken at random from the random lines (in an unlimited plane) which cut the area." This definition is intelligible enough, and Mr. Watson's solution of his own problem is in accordance with it.

The reasons which incline me to adhere to my first view of the most consistent interpretation of the phrase "a line drawn at random so as to cut a given area," are as follows:—In the first place, that the line cuts the area is to be taken as certainty. We cannot, at the outset, admit the supposition of the line being drawn at random in an unlimited plane, because it is contrary to the hypothesis in this problem that the line can fall anywhere else than across the area. Adopting the explanation that "at random" means "according to no law," it is evident that the "line drawn at random" in this question is not absolutely at random. The line in this question is restricted to the area, and is only to be drawn at random in all other respects; and it is in such a qualified sense only that the phrase can here be employed.

We will now examine in what way the line may be said to fall at random within the area. If an indefinite number of straight lines be distributed over a given area in certain arbitrary positions, and if one of these lines be selected at random, this last-named line cannot be said to have been drawn at random in the area unless the arbitrary positions from which it has been selected are such as would all be equally likely to occur if the matter were left to chance. Let $P(\theta)$ represent the perpendicular distance between two parallel straight lines drawn each at an inclination θ to some fixed line, so as to touch and enclose the area. If, then, a great number of straight lines be equally distributed over the area by equal increments of inclination, and also by placing the same invariable ultimate distance between every two adjacent parallel lines the number of parallel lines

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running across the figure at an inclination θ, will be expressed by $P(\theta)$ (using the infinitesimal unit.)

Now it has hitherto been assumed that a line drawn at random across the area will be equally likely to take up any one of the foregoing positions, and it has been inferred that the probability required is equal to the fraction whose numerator is the total possible number of restricted positions of the line and points, and denominator the total number of unrestricted positions. Let us notice a consequence of the assumption that all these positions of the line are equally probable. Since the line falls $P(\theta)$ times at the inclination θ, its tendency to fall at that inclination must be proportional to $P(\theta)$, a variable magnitude depending on the form of the contour. So far, then, from the angular position of the line being "according to no law" it seems that under this hypothesis it would be subject to a very remarkable law. If the area were a rectangle a mile in length and one foot wide, the tendency of the line to fall at right angles to the length is 5280 times its tendency to fall parallel to the length.

In seeking for a more consistent hypothesis, I would first observe that the assemblage of lines in the area from which a line is to be taken at random must itself be an assemblage of random lines. It must be the assemblage resulting from drawing a line at random in the area an indefinite number of times.* Hence the notion of randomness as belonging to the line precedes in order of thought the conception of a group of lines from which the random line may be regarded as having been selected haphazard; and therefore a definition of what constitutes randomness in the line ought to precede any attempt to define such group or assemblage of lines. Now it happens that of the two elements of position appertaining to the line, namely, inclination to a fixed line, and perpendicular distance from a fixed point, the first is the only one which is not of necessity restricted by the condition that the line is to cross a given area. The distance of the line from a fixed point must, under any hypothesis, be limited by the boundary of the area. The inclination of the line; on the contrary, is totally independent of the area, and need not be limited by it in any respect. By hypothesis everything is to be as far as possible according to no law; therefore the inclination of the line must be according to no law, for this is quite possible. The distance of the line from any fixed point being necessarily kept within limits, it only remains to leave that distance unrestricted at each inclination within those limits.

The effect will be that if a vast number of trials be made, the line will fall as often at one inclination as at another. At each inclination the same number of parallel lines will be scattered uniformly over the surface, and consequently their density at one inclination will be different from that at another, and will be inversely proportional to $P(\theta)$. There is nothing repugnant to the notion of chance in this result. A random agent frequently produces effects unequally distributed. Random shots from a gun strike the ground at points unequally distributed over the surface.

If now a number of straight lines be distributed equably over the area in the manner described at first, such lines will represent possible positions of the line drawn at random across the area, but all of these positions will not be equally probable. The probability of any one of them occurring will be $\frac{d\theta}{\pi} \cdot \frac{dp}{P(\theta)}$ where p is the distance of the line from a fixed point.

* In Professor Crofton's definition, quoted above, the lines there considered do not form such an assemblage. They are lines taken at random in an infinite plane, or (what is the same in effect) in any circle enclosing the area, instead of being lines taken at random *within the area*, which latter I propose to show constitute a different assemblage from the former.

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Let $Q(p, \theta)$ represent twice the product of the two areas into which the whole area Ω is divided by the line in this position. Then it follows that the probability of the line falling between the points at an inclination θ is $\frac{d\theta}{\pi}\int\frac{Q(p,\theta)}{\Omega^2}\frac{dp}{P(\theta)}$. The total probability, estimated in this way when the area is that of an equilateral triangle, is $=\frac{2}{15}+\frac{\sqrt{3}}{5\pi}=\cdot243599...$

If the line fall at the constant inclination of 60°, 75° or 90° to a side of the triangle, the probabilities calculated for these inclinations are

$$\frac{8}{30}=\cdot2666\ldots,\quad\frac{14-6\sqrt{3}}{15}=\cdot240513\ldots,\text{ and }\frac{7}{30}=\cdot2333\ldots$$

The value ·240513, calculated for an inclination of 75°, which is the mean of the greatest and least inclinations, approximates pretty closely to ·243599..., which is the value of the mean probability; and this was to be expected. On the hypothesis of random lines being scattered with equal density over the area at all inclinations, the probability in question takes the value $\frac{2}{15}+\frac{1}{10}\log_e 3=\cdot243194$, which differs very little from the former mean probability.

In conclusion, I would remark that it would be easy to obviate all uncertainty as to the hypothesis intended in problems of this sort by simply abandoning the use of such unqualified phrases as "a line drawn at random in a given area." For it is pretty clear that there is more than one at least plausible interpretation of such a phrase, and it costs but two or three words to specify in any case the particular kind of random line intended.

4067. (Proposed by Professor TOWNSEND, F.R.S.)—A uniform elastic beam, attached by its extremities to two fixed points A and B in the same horizontal line AB, and bent by its own weight W, is supported by a prop which keeps an intermediate point C in the same horizontal line with A and B; given all particulars, required the three pressures P, Q, R at the three points A, B, C.

Solution by the PROPOSER.

Denoting by a the semi-length of the beam, by E the modulus of elasticity of the material composing it, by I the moment of inertia of its transverse sectional area round the horizontal axis passing through its centre of gravity, and by $a+x$ and $a-x$ the distances of C from A and B respectively; then, since the evanescent deflection at C equals the sum of the opposite deflections due to W and to $-$R separately, the values of which are known to be

$$=\frac{1}{48}\cdot\frac{W}{EI}\cdot\frac{(5a^2-x^2)(a^2-x^2)}{a}\quad\text{and}\quad=-\frac{1}{6}\cdot\frac{R}{EI}\cdot\frac{(a^2-x^2)^2}{a},$$

respectively, consequently, equating their sum to 0 and striking out com-

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mon factors, we have at once $R = \frac{W}{8} \cdot \frac{5a^2 - x^2}{a^2 - x^2}$;

and as $P = \frac{1}{2}W - R \cdot \frac{a-x}{2a}$ and $Q = \frac{1}{2}W - R \cdot \frac{a+x}{2a}$,

therefore $P = \frac{W}{16} \cdot \frac{3a^2 + 8ax + x^2}{a(a+x)}$, $Q = \frac{W}{16} \cdot \frac{3a^2 - 8ax + x^2}{a(a-x)}$,

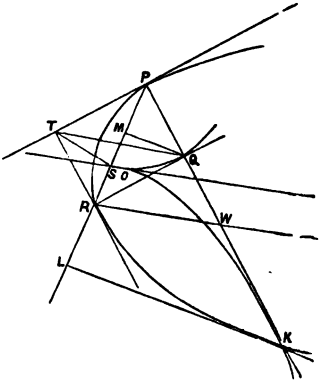
which are accordingly the three values required.

Corollary.—In the particular case when $x = 0$, that is, when the prop acts at the middle point of the beam, then $R = \frac{5}{8}W$, $P = Q = \frac{3}{16}W$, values which may be readily verified for that simple case.

4328. (Proposed by Professor WOLSTENHOLME.)—Prove that the tangent to the evolute of a parabola at the point where it cuts the parabola is also a normal to the evolute.

I. *Solution by* W. H. H. HUDSON, M.A.

Let KP be the tangent to the evolute at K, and normal to the parabola at P. Draw PSR through the focus, and let RT, RQ be the tangent and normal to the parabola at R; then TS is perpendicular to PR. Draw KL and QM perpendicular to PR; then if RW is parallel to the axis, it bisects PK; TQ bisects PR; therefore TQ is parallel to the axis and bisects PW; therefore PK = 4PQ, therefore PL = 4PM, therefore PS = 2PM; and from the equal triangles RTS, QPM, we have SR = PM, therefore RM = 2RS, therefore Q is the centre of curvature of R, and KQ is normal to the evolute at Q.



II. *Solution by the* PROPOSER.

The equation of a normal to the parabola $y^2 = 4ax$ is $y = m(x - 2a) - am^3$, and for the point where this touches the evolute $x - 2a - 3am^2 = 0$ and $y = 2am^3$; now suppose that the remaining point where this meets the evolute is $2a + 3am'^2$, $2am'^3$, then we shall have

$$2m'^3 = 3mm'^2 - m^3, \text{ or } (m' - m)^2(2m' + m) = 0,$$

or $m' = -\frac{1}{2}m$, whence if the straight line be both tangent and normal to the evolute, $\frac{m}{2} = \frac{1}{m}$, or $m^2 = 2$; the point of contact is $8a$, $\pm 4a\sqrt{2}$, which

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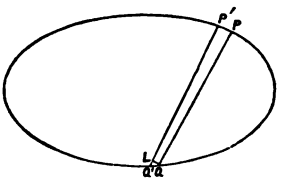
lies on the parabola. Taking the upper sign, the straight line is normal to the parabola at the point $2a, -2a\sqrt{2}$; and to the evolute at the point $\frac{1}{2}a, \frac{1}{2}a\sqrt{2}$. The straight line is one of the minimum normal chords of the parabola, its whole length being $6a\sqrt{3}$, and it is of course the maximum chord intercepted between the parabola and its evolute, the length of this chord being one-fourth the former, or $\frac{3}{2}a\sqrt{3}$.



[Concerning minimum normal chords in general, if PQ, P′Q′ be two adjacent chords normal at P, P′, and QL be drawn perpendicular to P′Q′, then when PQ is a minimum, Q′L must vanish, which can only happen in two ways, (1) when Q is the centre of curvature at P, so that Q′ coincides with Q (to the first order of small quantities), (2) when PQ is a normal both at P and Q, so that L coincides with Q′ (to the first order).

Thus in an ellipse of eccentricity $> \frac{1}{\sqrt{2}}$, as P moves round the curve we shall have four maximum normal chords (each axis twice over), and four minimum normal chords, when Q comes to the points where the evolute crosses the ellipse. So also in the curve $(x^2+y^2-4a^2)^3+27a^2y^4=0$ (an involute of the four cusped hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=(2a)^{\frac{2}{3}}$ passing through a cusp), which appears very like an ellipse with axes $4a$, $2a$, these are the maximum normal chords $4a$, $2a$, normal at both ends, and four minimum normal chords of length $\frac{6a}{2+2^{\frac{1}{3}}}$ or $1{\cdot}919964a$, the minimum normal chords in the ellipse whose axes are $4a, 2a$ being $\frac{12\sqrt{3}}{5\sqrt{5}}a$ or $1{\cdot}859037a$, and the angles which these minimum chords make with the axis of x are respectively $51°\ 33'\ 38''{\cdot}6$ and $53°\ 47'\ 55''{\cdot}2$.]

4214. (Proposed by E. B. Seitz.)—The first of two casks contains a gallons of wine, and the second contains b gallons of water. c gallons are drawn from the second cask, and then c gallons are drawn from the first cask and poured into the second, and the deficiency in the first supplied with c gallons of water. Required the quantity of wine in the second cask after n such operations as the one described above.

I. *Solution by* Walter Siverly.

Let u_n and v_n be the wine in the first and second casks respectively at the end of the nth operation; then we have

$$u_n + \frac{c}{a}u_n = u_{n+1}, \text{ or } u_{n+1} - \left(1-\frac{c}{a}\right)u_n = 0.$$

Integrating, we have $u_n = c_1\left(1-\frac{c}{a}\right)^n$;

and when $n=0$, $u_n = a$, therefore $c_1 = a$, and $u_n = a\left(1-\frac{c}{a}\right)^n$,

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$$v_n - \frac{c}{b} v_n + \frac{c}{a} u_n = v_{n+1}, \text{ or } v_{n+1} - \left(1 - \frac{c}{b}\right) v_n = c\left(1 - \frac{c}{a}\right)^n.$$

Integrating, $v_n = c_2\left(1 - \frac{c}{b}\right)^n + \frac{ab}{a-b}\left(1 - \frac{c}{a}\right)^n$;

and when $n = 0$, $u_n = 0$, therefore $c_2 = -\frac{ab}{a-b}$;

hence $v_n = \frac{ab}{a-b}\left\{\left(1 - \frac{c}{a}\right)^n - \left(1 - \frac{c}{b}\right)^n\right\}$.

II. *Solution by* ARTEMAS MARTIN.

Let $Q_1, Q_2, Q_3, \ldots Q_n$ represent the quantity of *wine* in the second cask after 1, 2, 3, ... n operations, and put $\left(1 - \frac{c}{a}\right) = p$, $\left(1 - \frac{c}{b}\right) = q$; then we have

$$Q_1 = c = c\left(\frac{p-q}{p-q}\right), \quad Q_2 = c\left(\frac{p^2-q^2}{p-q}\right), \quad Q_3 = c\left(\frac{p^3-q^3}{p-q}\right) \ldots\ldots$$

$$Q_n = c\left(\frac{p^n-q^n}{p-q}\right) = \frac{b\left[\left(1 - \frac{c}{a}\right)^n - \left(1 - \frac{b}{a}\right)^n\right]}{\left(1 - \frac{b}{a}\right)}.$$

[When $b = a$, $Q_n = nc\left(1 - \frac{c}{a}\right)^{n-1}$].

4392. (Proposed by S. ROBERTS, M.A.)—If N_p denotes the number of terms in a symmetrical determinant of p rows and columns; show that the successive numbers are given by the equation

$N_k - N_{k-1} - (k-1)^2 N_{k-2} + \frac{1}{2}(k-1)(k-2)\{N_{k-3} + (k-3) N_{k-4}\} = 0$,

k being positive and N_0 being taken equal to unity.

I. *Solution by* Professor CAYLEY, F.R.S.

It is a curious coincidence that the question of determining the number of distinct terms in a symmetrical determinant has been recently solved by Captain Allan Cunningham in a paper in the last number of the *Quarterly Journal of Science*;* and the question having been proposed to me by Mr. Glaisher, I have also solved it in a paper printed in the April Number of the *Monthly Notices of the Royal Astronomical Society*. I there

obtain $N_k = 1 \cdot 2 \ldots k$ coeff. x^k in $\frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{(1-x)^{\frac{1}{2}}}$,

viz., writing $u = N_0 + N_1 \frac{x}{1} + N_2 \frac{x^2}{1 \cdot 2} + \ldots$,

* I have not the volume at hand to refer to, but he obtains an equation of differences, and gives the numbers 1, 2, 5, 73, 393 (should be 388) ...

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I show that u satisfies the differential equation

$$2\frac{du}{dx} = \left\{1 + x + \frac{1}{1-x}\right\} u,$$

giving when the constant is determined $u = \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{(1-x)^{\frac{1}{2}}}$. Writing the differential equation in the form $2(1-x)\frac{dy}{dx} = (2-x^2)\,u$,

we at once obtain for N_k the equation of differences

$$N_k - kN_{k-1} + \tfrac{1}{2}(k-1)(k-2)\,N_{k-3} = 0,$$

which is in fact a particular first integral of Mr. Roberts's equation; viz., from the above equation we have

$$N_{k-1} - (k-1)\,N_{k-2} + \tfrac{1}{2}(k-2)(k-3)\,N_{k-4} = 0,$$

and multiplying this last by $k-1$ and adding, we have

$$N_k - N_{k-1} - (k-1)^2 N_{k-2} + \tfrac{1}{2}(k-1)(k-2)\left\{N_{k-3} + (k-3)\,N_{k-4}\right\} = 0,$$

which is the equation obtained by Mr. Roberts. It thence appears that the *general* first integral of his equation is

$$N_k - kN_{k-1} + \tfrac{1}{2}(k-1)(k-2)\,N_{k-3} = (-)^k C\,1.2\ldots(k-1).$$

The equation $N_k = kN_{k-1} - \frac{1}{2}(k-1)(k-2)\,N_{k-3}$

gives very readily the numerical values, viz.,

$1 = 1.1-0$	$17 = 4.5 - 3.1$	$2461 = 7.388 - 15.17$
$2 = 2.1-0$	$73 = 5.17 - 6.2$	$18155 = 8.2461 - 21.73.$
$5 = 3.2-1.1$	$388 = 6.73 - 10.5$	

II. *Solution by the* Proposer.

The determinant is of the order k in the elements, k being the number of rows and columns. Moreover if the elements are distinguished by suffixes in the usual way, thus—

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & \cdots & a_{1,k} \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & \cdots & a_{2,k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k,1}, & a_{k,2}, & a_{k,3}, & \cdots & a_{k,k} \end{vmatrix}$$

each of the suffixes occurs twice in each term, and the problem is to determine the number of distinct ways in which $2k$ things, two of a sort, can be made up into parcels of two.

The number of ways, (11) being one of the parcels, is N_{k-1}.

The number of ways, (12), (12) being two of the parcels, is N_{k-2}, and there are $k-1$ of these systems, containing (13) (13), (14) (14), &c.

The number of ways, (12), (13), (23) being three of the parcels, is N_{k-3}, and there are $\frac{1}{2}(k-1)(k-2)$ such systems.

The number of ways, (12), (13), (24), (34) being four of the parcels, is N_{k-4}, and there are $\frac{1}{2}(k-1)(k-2)(k-3)$ such systems.

The number of ways, (12), (13), (24), (35), (45) being five of the parcels, is N_{k-5}, and there are $\frac{1}{2}(k-1)(k-2)(k-3)(k-4)$ such systems, and so on.

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Hence we have

$$N_k = N_{k-1} + (k-1)\,N_{k-2} + \tfrac{1}{2}(k-1)(k-2)\left\{N_{k-3} + (k-3)\,N_{k-4} + (k-3)(k-4)\,N_{k-5} + \&c.\right\},$$

taking $N_0 = 1$.

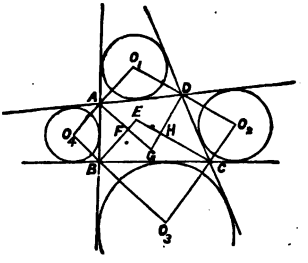
From this we obtain immediately

$$N_k = N_{k-1} + (k-1)^2 - \tfrac{1}{2}(k-1)(k-2)\left\{N_{k-3} + (k-3)\,N_{k-4}\right\},$$

an equation of finite differences which it would be interesting to solve.

4383. (Proposed by W. B. Davis, B.A.)—If the sides of a quadrilateral be produced, and four circles are inscribed, prove that the centres of these circles lie on the circumference of another circle.

Solution by Belle Easton; A. B. Evans, M.A.; *and others.*



If we bisect the four angles of the quadrilateral ABCD, these bisectors will form a second quadrilateral EFGH inscriptible in a circle; for, denoting a right angle by R, we have, in the triangles ADG and BEC,

$\frac{1}{2}A + \frac{1}{2}D + G = 2R$, $\frac{1}{2}B + \frac{1}{2}C + E = 2R$;

$\therefore\ \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D + E + G = 4R$,

and therefore $E + G = 2R$.

Therefore E and G are supplements of each other, and the quadrilateral EFGH is inscriptible in a circle.

Now the angles O_1, O_2, O_3, O_4 are respectively supplemental to G, H, E, F; and therefore the quadrilateral $O_1O_2O_3O_4$ is inscriptible in a circle.

4391. (Proposed by Professor Wolstenholme, M.A.)—Prove that

$$\begin{vmatrix} a, & 1, & 0, & 0, & 0, \ldots\ldots & 0 \\ 1, & a, & 1, & 0, & 0, \ldots\ldots & 0 \\ 0, & 1, & a, & 1, & 0, \ldots\ldots & 0 \\ \ldots\ldots & & & & & \\ 0, & 0, & \ldots\ldots & 1, & a, & 1 \\ 0, & 0, & \ldots\ldots & 0, & 1, & a \end{vmatrix} \equiv \frac{\sin(n+1)\theta}{\sin\theta}, \text{ where } 2\cos\theta = a\text{; and also}$$

$$\equiv a^n - (n-1)\,a^{n-2} + \frac{(n-2)(n-3)}{\lfloor 2}\,a^{n-4} - \ldots$$

I. *Solution by* J. W. L. Glaisher, M.A.

Denoting the determinant by P_n, we have by development

$$P_n = aP_{n-1} - P_{n-2},$$

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whence $$\frac{P_n}{P_{n-1}} = a - \frac{1}{\frac{P_n}{P_{n-2}}} = a - \cfrac{1}{a - \cfrac{1}{a \ldots (n-1 \text{ times})}}.$$

Also $$\sin(n+1)\theta + \sin(n-1)\theta = 2\sin n\theta \cos\theta,$$

whence $$\frac{\sin(n+1)\theta}{\sin n\theta} = a - \frac{1}{\frac{\sin n\theta}{\sin(n-1)\theta}} = a - \cfrac{1}{a - \cfrac{1}{a \ldots (n-1 \text{ times})}}.$$

Thus $$\frac{P_n}{P_{n-1}} = \frac{\sin(n+1)\theta}{\sin n\theta},$$

and therefore $$\frac{P_n}{P_1} = \frac{\sin(n+1)\theta}{\sin 2\theta}; \text{ whence } P_n = \frac{\sin(n+1)\theta}{\sin\theta}.$$

The equivalence of the series and determinant can be shown in the same way, viz., by proving that the former satisfies the relation

$$P_n = aP_{n-1} - P_{n-2}.$$

It may be remarked that the value of $\frac{P_n}{P_{n-1}}$ as a continued fraction can be easily obtained, without developing the determinant, by observing that $x_1 = \frac{P_n}{P_{n-1}}$ follows from the system of equations

$$ax_1 + x_2 = 1, \quad x_1 + ax_2 + x_3 = 0, \quad \ldots\ldots, \quad x_{n-1} + ax_n = 0.$$

II. *Solution by the* Proposer.

For this determinant is the resultant of the equations

$$ax_1 + x_2 = 0, \quad x_1 + ax_2 + x_3 = 0, \quad x_2 + ax_3 + x_4 = 0, \ \ldots\ldots,$$
$$x_{n-1} + a_1 x_{n-1} + x_n = 0, \quad x_{n-1} + ax_n = 0,$$

and since the equation connecting any three is $x_{r-1} + ax_r + x_{r+1} = 0$, we must have $$x_r = A\alpha^{r-1} + B\beta^{r-1};$$

where α, β are the roots of the equation $z^2 + az + 1 = 0$;

therefore $$x_1 = A + B, \quad x_2 = -ax_1 = A\alpha + B\beta;$$

therefore $$A = \frac{\beta x_1 - x_2}{\beta - \alpha} = x, \quad \frac{\beta + \alpha}{\beta - \alpha} \equiv \frac{ax_1}{\alpha - \beta}, \text{ for } \alpha + \beta = -a.$$

So $$B = \frac{\beta x_1}{\beta - \alpha}, \text{ and } x = x_1 \frac{\alpha^r - \beta^r}{\alpha - \beta},$$

whence $$x_n = \frac{x_1}{\alpha - \beta}(\alpha^n - \beta^n), \text{ and } x_{n-1} = \frac{x_1}{\alpha - \beta}(\alpha^{n-1} - \beta^{n-1}),$$

and $$x_{n-1} + ax_n = 0,$$

therefore the resultant is $$\frac{a(\alpha^n - \beta^n) + \alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}.$$

But if we put $$a = 2\cos\theta, \quad \alpha = -\cos\theta + \sqrt{(-1)}\sin\theta,$$
$$\beta = -\cos\theta - \sqrt{(-1)}\sin\theta,$$

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whence the resultant is

$$\frac{2\cos\theta\left\{2\sqrt{(-1)}\sin n\theta\right\} - 2\sqrt{(-1)}\sin(n-1)\theta}{2\sqrt{(-1)}\sin\theta} \text{ or } \frac{\sin(n+1)\theta}{\sin\theta} \text{ (to a factor).}$$

To express this in powers of a, we have

$$\frac{1}{1+2x\cos\theta+x^2} = 1-x(a+x)+x^2(a+x^2-x^3(a+x)^3+\ldots\ldots,$$

and also $$= \frac{1}{a-\beta}\left(\frac{a}{1-ax}-\frac{\beta}{1-\beta x}\right) = \frac{1}{a-\beta}\left\{a(1+ax+a^2x^2+\ldots)\right\}$$
$$-\beta(1+\beta x+\beta^2x^2+\ldots),$$

in which the coefficient of x^n is $\frac{a^{n+1}-\beta^{n+1}}{a-\beta}$ or $(-1)^n\frac{\sin(n+1)\theta}{\sin\theta}$.

But the coefficient of x^n from the other expansion is

$$(-1)^n\left\{a^n-(n-1)a^{n-2}+\frac{(n-2)(n-3)}{\lfloor 2}a^{n-4}+\ldots\ldots\right\};$$

whence $$\frac{\sin(n+1)\theta}{\sin\theta} = a^n-(n-1)^{n-2}+\frac{(n-2)(n-3)}{\lfloor 2}a^{n-4}-\ldots\ldots$$

Since it is manifest that in the determinant the coefficient of a^n is 1, we see that either of these equivalents is the value of the determinant.

4395. (Proposed by J. W. L. Glaisher, B.A.)—Prove that, n being a positive integer,

$$\int_0^1\int_0^\pi x^{-\rho x\cos\theta}\cos(\rho x\log x\sin\theta+n\theta)\,dx\,d\theta = \frac{\pi\rho^n}{(n+1)^{n+1}}.$$

I. *Solution by* F. D. Thomson, M.A.

Let $u = \int_0^\pi \epsilon^{-a\cos\theta}\cos(a\sin\theta+n\theta)\,d\theta$, where n is a positive integer;

$$\frac{du}{da} = -\int_0^\pi \epsilon^{-a\cos\theta}\cos\left\{a\sin\theta+(n-1)\theta\right\}d\theta, \quad \ldots\ldots,$$

$$\frac{d^nu}{da^n} = (-)^n\int_0^\pi \epsilon^{-a\cos\theta}\cos(a\sin\theta)\,d\theta$$

$$= (-)^n\int_0^\pi\left(1-a\cos\theta+\frac{a^2}{\lfloor 2}\cos 2\theta-\&c.\right) = (-)^n\pi \ldots\ldots\ldots (1).$$

Also $$\frac{d^ru}{da^r}_{a=0} = (-)^r\int_0^\pi\cos(n-r)\theta\,d\theta = 0, \quad (r<n) \ldots\ldots\ldots (2),$$

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therefore integrating (1) and remembering (2), we have $u = (-)^n \frac{\pi a^n}{\lfloor n}$;

therefore

$$\int_0^1 \int_0^\pi \epsilon^{-\rho x \log x \cos\theta} \cos(\rho x \log x \sin\theta + n\theta)\, d\theta = (-)^n \frac{\pi}{\lfloor n} \rho^n \int_0^1 x^n (\log x)^n\, dx$$

$$= (-)^n \frac{\pi \rho^n}{\lfloor n} \times (-)^n \frac{\lfloor n}{(n+1)^{n+1}} = \frac{\pi \rho^n}{(n+1)^{n+1}}. \quad \text{(HYMERS' } \textit{Integral Calculus}\text{, p. 130.)}$$

II. *Solution by the* PROPOSER.

From the definition of the gamma-function,

$$\int_0^\infty e^{-nx} x^{n+1}\, dx = \frac{\Gamma(n)}{n^n}; \text{ whence } \int_0^\infty e^{-x+pxe^{-x}}\, dx = \Sigma_1^\infty \frac{p^{n-1}}{n^n}.$$

Transforming this last integral there results

$$\int_0^1 x^{-px}\, dx = \Sigma_1^\infty \frac{p^{n-1}}{n^n}.$$

Herein replace p by $\rho(\cos\theta + i \sin\theta)$, and

$$\int_0^1 x^{-\rho x \cos\theta} \cos(\rho x \log \sin x \sin\theta)\, dx = \Sigma \frac{\rho^{n-1} \cos(n-1)\theta}{n^n},$$

$$-\int_0^1 x^{-\rho x \cos\theta} \sin(\rho x \log \sin x \sin\theta)\, dx = \Sigma \frac{\rho^{n-1} \sin(n-1)\theta}{n^n}.$$

Multiply the first of these two integrals by $\cos n\theta$ and the second by $\sin n\theta$, integrate both with regard to θ between the limits 0 and π, and add the results. We thus obtain the double integral in question. By subtraction we should have found

$$\int_0^1 \int_0^\pi x^{-\rho x \cos\theta} \cos(\rho x \log x \sin\theta - n\theta)\, dx\, d\theta = 0.$$

3958. (Proposed by Professor WOLSTENHOLME.)—1. The squares of the semi-axes of the conic of closest contact at any point of a curve are the values of x given by the equation

$$x^2(9 + b^2 - 3ac)^3 - 9a^2(18 + 2b^2 - 3ac)(9 + b^2 - 3ac)\, x + 729a^4 = 0,$$

and the product of the focal distances is $9a^2 \div (9 + b^2 3ac)$; a, b, c being the values of ρ, $\frac{d\rho}{ds}$, $\frac{d^2\rho}{ds^2}$ at the point, where ρ is the radius of curvature and s the arc.

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2. Hence prove that the eccentricity e is given by the equation

$$9\frac{(e^2-2)^2}{1-e^2} = \frac{(18+2b^2-3ac)^2}{9+b^2-3ac};$$

and deduce the conditions that the conic may be a circle.

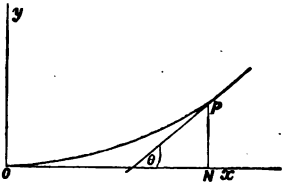
3. The equation of the conic of closest contact at any point of a given curve referred to the tangent and normal at the point as co-ordinate axes is $Ax^2+2Bxy+Cy^2=2y$; the values of A, B, C being

$$\frac{1}{\rho},\quad -\frac{1}{3\rho}\frac{d\rho}{ds},\text{ and }\frac{1}{\rho}+\frac{2}{9\rho}\left(\frac{d\rho}{ds}\right)^2-\frac{1}{3}\frac{d^2\rho}{ds^2},$$

where ρ is the radius of curvature at the point, and s the arc to that point from some fixed point.

Solution by the PROPOSER.

If x, y be coordinates of a point P, referred to the tangent and normal at O, and θ the angle which the tangent at P makes with the axis of x, then



$$\frac{dx}{ds}=\cos\theta,\quad \frac{dy}{ds}=\sin\theta,\quad \frac{ds}{d\theta}=\rho;$$

therefore $\dfrac{d^2x}{ds^2}=-\dfrac{\sin\theta}{\rho},\ \dfrac{d^2y}{ds^2}=\dfrac{\cos\theta}{\rho},$

$$\frac{d^3x}{ds^3}=-\frac{1}{\rho^2}\cos\theta+\frac{1}{\rho^2}\frac{d\rho}{ds}\sin\theta,\quad \frac{d^3y}{ds^3}=-\frac{1}{\rho^2}\sin\theta-\frac{1}{\rho^2}\frac{d\rho}{ds}\cos\theta,\ \&c.;$$

whence if a, b, c denote the values of ρ, $\dfrac{d\rho}{ds}$, and $\dfrac{d^2\rho}{ds^2}$ at the origin, we get

$$x=s-\frac{s^3}{6a^2}+\frac{s^4}{8a^3}b+\ldots,\quad y=\frac{s^2}{2a}-\frac{s^3}{6a^2}b-\frac{s^4}{24a^3}(1-2b^2+ac)+\ldots.$$

Hence if $Ax^2+2Bxy+Cy^2=2y$, be the equation of the conic of closest contact, substituting these values for x, y and equating coefficients of the different powers of s up to s^4, which gives five-pointic contact, we get

$$A=\frac{1}{a},\quad \frac{B}{a}=-\frac{1}{3a^2}b \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1,2),$$

$$-\frac{A}{3a^2}-\frac{B}{3a^2}b+\frac{C}{4a^2}=-\frac{1}{12a^3}(1-2b^2+ac);\text{ or }C=\frac{1}{a}+\frac{2}{9a}b^2-\frac{1}{8}c\ldots(3).$$

But in the conic $Ax^2+2Bxy+Cy^2=2y$, the centre is given by the equations $\quad Ax_0+By_0=0,\quad Bx_0+Cy_0=1$; or $\dfrac{x_0}{B}=\dfrac{y_0}{-A}=\dfrac{1}{B^2-AC}$,

and the equation of the conic referred to its centre is

$$Ax^2+2Bxy+Cy^2=y_0=\frac{A}{AC-B^2};$$

whence the equation giving the squares of the semi-axes is

$$\left(\frac{A}{AC-B^2}-Ax\right)\left(\frac{A}{AC-B^2}-Cx\right)=B^2x^2,$$

or $$x^2(AC-B^2)^3-xA(A+C)(AC-B^2)+A^2=0.$$

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But $AC-B^2 = \frac{1}{9a^2}(9+b^2-3ac)$; $A+C = \frac{1}{9a}(18+2b^2-3ac)$,

and the equation becomes

$$x^2(9+b^2-3ac)^3 - 9a^2(9+b^2-3ac)(18+2b^2-3ac)x + 729a^4 = 0.$$

The product of focal distance = sum of squares of semi-axes $= -x_0^2 - y_0^2$

$$= \frac{A(A+C)}{(AC-B^2)^2} - \frac{A^2+B^2}{(AC-B^2)^2} = \frac{1}{AC-B^2} = \frac{9a^2}{9+b^2-3ac}.$$

For the eccentricity $\frac{(e^2-2)^2}{1-e^2} = \frac{(A+C)^2}{AC-B^2} = \frac{(18+2b^2-3ac)^2}{9(9+b^2-3ac)}$;

and when $e=0$, this gives $(2b^2-3ac)^2 + 36b^2 = 0$, whence generally for the conic of closest contact to be a circle, b and $c=0$, or $\frac{d\rho}{ds}$ and $\frac{d^2\rho}{ds^2} = 0$, as is obvious *à priori*.

The above equations for x, y are very useful in all questions of contact, it would be very easy to carry the approximations farther, but not often to any purpose.

The following results easily follow:—if PO, OQ be two tangents at contiguous points, the limiting value of $\frac{1}{PO} \sim \frac{1}{OQ}$ is $\frac{2b}{3a}$; the limiting value of $\frac{PO+OQ-\text{arc PQ}}{PO+OQ-\text{chord PQ}} = \frac{2}{3}$, or $\frac{OP+OQ-\text{arc PQ}}{\text{arc PQ}-\text{chord PQ}} = 2$;

if equal small arcs δs be measured in the same direction from a point along a curve and along the parabola of closest contact, the distance between their extremities is ultimately $\frac{\delta s^4}{24a^3}(3+\frac{1}{2}b^2-ac)$; and if equal small arcs δs be measured along the curve and the conic of closest contact, the distance between their extremities will be ultimately

$$\frac{\delta s^5}{120a^3}\left(ad-bc+\frac{4b}{9a}(9+b^2)\right),$$

where d denotes $\frac{d^3\rho}{ds^3}$ in the curve. (If we put this $= 0$ we obtain the general differential intrinsic equation of all conics). As far as I have carried the approximation I make the results

$$PO = \frac{s}{2} - \frac{s^2}{12\rho}\frac{d\rho}{ds} + \frac{s^3}{24\rho^2}\left\{1+\left(\frac{d\rho}{ds}\right)^2 - \rho\frac{d^2\rho}{ds^2}\right\} + \ldots,$$

$$OQ = \frac{s}{2} + \frac{s^2}{12\rho}\frac{d\rho}{ds} + \frac{s^3}{24\rho^2}\left\{1-\left(\frac{d\rho}{ds}\right)^2 + \rho\frac{d^2\rho}{ds^2}\right\} + \ldots, \quad \text{arc PQ} = s,$$

$$\text{chord PQ} = s - \frac{s^3}{24\rho^2} + \frac{s^4}{24\rho^3}\frac{d\rho}{ds} - \frac{s^5}{120\rho^4}\left\{\tfrac{1}{16}-\tfrac{1}{8}\left(\frac{d\rho}{ds}\right)^2 + \tfrac{3}{2}\rho\frac{d^2\rho}{ds^2}\right\} + \ldots.$$

If PQ be a chord parallel to the tangent at O, the straight line bisecting the external angle between PO, OQ, will meet PQ in a point whose distance from O is ultimately $6\rho\frac{ds}{d\rho}$ (*Sen. House*, 1869). If C be the middle point of the chord PQ, OC makes ultimately with the normal at O the angle $\tan^{-1}\left(\frac{1}{8}\frac{d\rho}{ds}\right)$ (*Smith's Prize, Examination*, 1869.) The latus-rectum

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of the parabola of closest contact is $54\rho \div \left\{9 + \left(\frac{d\rho}{ds}\right)^2\right\}^{\frac{1}{2}}$ (*Senate House*, 1872). The angles which the axes of the conic of closest contact make with the tangent at any point of a curve are given by the equation $\tan 2\theta = \frac{6b}{2b^2 - 3ac}$. The length of the axis of the cycloid of closest contact is $\frac{1}{4}a(1+b^2)^{\frac{1}{2}}$.

4317. (Proposed by the EDITOR.)—Prove that the sum of the squares of the radius of curvature at any point of a cycloid, and of the arc from that point to the vertex of the curve, is constant; and extend this property to epicycloids.

Solution by R. TUCKER, M.A.

Using the Notation of sections 9 and 17 of my article on Radial Curves (*Reprint*, Vol. I., pp. 16—19), we have

$$s = 4a \sin \Theta, \quad \rho = 4a \cos \Theta,$$

for the cycloid; whence we have at once (as is shown in Art. 18 of the above quoted article) $\rho^2 + s^2 = 16a^2 =$ constant.

In the case of Epi- and Hypo-cycloids, we have

$$s = l \sin n\phi, \quad \rho = nl \cos n\phi,$$

and, therefore, $n^2 s^2 + \rho^2 = n^2 l^2$.

(n is > or < unity, according as the curve is a Hypo- or Epi-cycloid.)

[See also Dr. Booth's *New Geometrical Methods*, Art. 209.]

4279. (Proposed by A. W. PANTON, M.A.)—Prove that the sum of the areas of the two ovals of a Cartesian is equal to twice the area of the circle whose centre is the triple focus, and which passes through the points of contact of the double tangent.

Solution by SAMUEL ROBERTS, M.A.

I have mentioned this result in a paper on Cartesians in the *Proceedings of the London Mathematical Society*. The equation of the curve being

$$\rho^2 - 2(A + B\cos\omega)\rho + C^2 = 0,$$

we have, ρ_1 and ρ_2 being the roots,

$$\tfrac{1}{2}(\rho_1^2 + \rho_2^2) = 2(A + B\cos\omega)^2 - C^2 = (B^2 + 2A^2 - C^2) + 4AB\cos\omega + B^2\cos 2\omega$$

therefore $\int \frac{\rho_1^2 + \rho_2^2}{2} d\omega = (B^2 + 2A^2 - C^2)\omega + 4AB\sin\omega + \frac{1}{4}B^2 \sin 2\omega.$

The origin is the extreme inner focus, and the limits are 0 and 2π.

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4276. (Proposed by J. J. WALKER, M.A.)—If h_1, h_2, h_3 are the depths of the corners A, B, C of a plane triangle below the surface of water; prove that the depth (k) of the centre of pressure (K) is given by

$$k = \frac{h_1^2 + h_2^2 + h_3^2 + h_2h_3 + h_3h_1 + h_1h_2}{2(h_1 + h_2 + h_3)}.$$

I. *Solution by the* PROPOSER.

Suppose A to be the highest corner, C the lowest. Produce BA to meet the surface in D, and join CD. For the centre of pressure (K′) on the area ADC the construction is obvious. Thus, draw AA′, CC′ perpendicular to the plane of the triangle, and equal to h_1, h_3 respectively; join DA′, A′C′, C′D. Let a perpendicular through the centre of gravity of AA′C′C on AC meet it in E, and A′C′ in E′; join DE and on it take $DK' = \frac{3}{4}DE$. Let the depth of K′ be k'. Then we have

$$k' = \tfrac{3}{4}EE' = \frac{3\{(2h_1 + h_3)h_1 + (2h_3 + h_1)h_3\}}{4 \times 3(h_1 + h_3)} = \frac{h_1^2 + h_3^2 + h_1h_3}{2(h_1 + h_3)}.$$

Similarly if k'' be the depth of the centre of pressure (K″) on the area BDC, $k'' = \frac{h_2^2 + h_3^2 + h_2h_3}{2(h_2 + h_3)}$. Let P, P′, P″ be the pressures on the areas ABC, ADC, BDC respectively. These areas are proportional to $h_2 - h_1$, h_1, h_2; consequently

$$P : P' : P'' = (h_2 - h_1)(h_1 + h_2 + h_3) : h_1(h_1 + h_3) : h_2(h_2 + h_3).$$

But $kP = k''P'' - k'P'$, therefore

$$k(h_2 - h_1)(h_1 + h_2 + h_3) = \tfrac{1}{2}\{(h_2^2 + h_3^2 + h_2h_3)h_2 - (h_1^2 + h_3^2 + h_1h_3)h_1\}$$
$$= \tfrac{1}{2}\{h_2^3 - h_1^3 + (h_2^2 - h_1^2)h_3 + (h_2 - h_1)h_3^2\};$$

therefore
$$k = \frac{h_1^2 + h_2^2 + h_3^2 + h_2h_3 + h_3h_1 + h_1h_2}{2(h_1 + h_2 + h_3)}.$$

Suppose d, e, p, x, x', x'' to be the distances of D, E, A, K, K′, K″ respectively from BC. Then we have

$$4x' = d + 3e = \left(\frac{h_2}{h_2 - h_1} + \frac{2h_1 + h_3}{h_1 + h_3}\right)p = \frac{h_2(h_1 + h_3) + (h_2 - h_1)(2h_1 + h_3)}{(h_2 - h_1)(h_1 + h_3)}p.$$

Also
$$4x'' = d = \frac{h_2}{h_2 - h_1}p.$$

Now $4xP = 4x''P'' - 4x'P'$, or

$$4x(h_2 - h_1)(h_1 + h_2 + h_3) = \frac{h_2^2(h_2 + h_3) - h_1h_2(h_1 + h_3) - h_1(h_2 - h_1)(2h_1 + h_3)}{h_2 - h_1}p$$
$$= \{h_2(h_1 + h_2 + h_3) - h_1(2h_1 + h_3)\}p = (h_2 - h_1)(2h_1 + h_2 + h_3)p;$$

whence
$$x = \frac{2h_1 + h_2 + h_3}{4(h_1 + h_2 + h_3)}p;$$

which agrees with the formula given by Professor Curtis, *Messenger of Mathematics*, Vol. ii.

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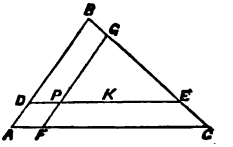
II. *Solution by* ASHER B. EVANS, M.A.



Let P be any point in the triangle ABC; through P draw DE and FG parallel to AC and AB. Let d_1, e_1, p_1 be the depths of D, E, P beneath the surface of water, and put AF $= x$, FP $= y$. Then, evidently,

$$AB \cdot d_1 = BD \cdot h_1 + AD \cdot h_2,$$

$$BC \cdot e_1 = BE \cdot h_3 + CE \cdot h_2,$$

and $$DE \cdot p_1 = PE \cdot D_1 + DP \cdot E_1;$$

whence, if a, b, c be the sides of the triangle ABC, we have

$$p_1 = (h_2 - h_1)\frac{y}{c} + (h_3 - h_1)\frac{x}{b} + h_1 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

When the point P coincides with the centre of pressure (K), p_1, x, y become k, x_0, y_0, and we have

$$x_0 = \frac{\iint x\,(1)\,dx\,dy}{\iint (1)\,dx\,dy} = \frac{\frac{1}{2}\frac{1}{3}b^2c\,(h_1 + h_2 + 2h_3)}{\frac{1}{6}bc\,(h_1 + h_2 + h_3)} = \frac{1}{2}b\left(1 + \frac{h_3}{h_1 + h_2 + h_3}\right),$$

$$y_0 = \frac{\iint y\,(1)\,dx\,dy}{\iint (1)\,dx\,dy} = \frac{\frac{1}{2}\frac{1}{3}c^2b\,(h_1 + 2h_2 + h_3)}{\frac{1}{6}bc\,(h_1 + h_2 + h_3)} = \frac{1}{2}c\left(1 + \frac{h_2}{h_1 + h_2 + h_3}\right),$$

since the limits of x are 0 and $b\left(1 - \frac{y}{c}\right) = DE$; and those of y are 0 and $c = AB$. For the depth of the centre of pressure, we have, from (1),

$$k = (h_2 - h_1)\frac{y_0}{c} + (h_3 - h_1)\frac{x_0}{b} + h_1 = \frac{h_1^2 h_2^2 + h_3^2 + h_2h_3 + h_3h_1 + h_1h_2}{2\,(h_1 + h_2 + h_3)}.$$

4242. (Proposed by S. ROBERTS, M.A.)—The two ovals of a Cartesian meet the axis in four real points A, B, C, D, and in the same succession as the letters. The axial diameters may therefore be grouped in three pairs [AB, CD], [AC, BD], [AD, BC]. If three ellipses are drawn with their principal semi-diameters equal to the three pairs of diameters of the Cartesian, then periphery of outer oval $= \frac{1}{2}$ sum of peripheries of ellipses on [AC, BD], [AB, CD], periphery of inner oval $= \frac{1}{2}$ difference of same, and the arcs of the ovals intercepted by the axis and the apices, points of contact of tangents from the exterior focus, are expressible in terms of arcs of the third ellipse and the peripheries of the first ellipses. In the case of a nodal limaçon, the loop periphery $= \frac{1}{2}$ (periphery of ellipse whose principal semi-diameters are equal to the diameters of the loops $\pm$ four times the difference of the loop diameters).

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Solution by the PROPOSER.

In a paper on Cartesians, printed in the volume for 1870 of the *Proceedings of the Mathematical Society*, I remarked that the lengths of the ovals of a Cartesian are expressed by syzygetic relations between two elliptic quadrants. I have since worked out the question as follows:—

If $A > B$ and $A - B > C$, the equation $\rho^2 - 2(A + B\cos\omega)\rho + C^2 = 0$ represents a Cartesian the extreme inner focus being the origin. Corresponding points lie in this case on adjacent parts of the ovals.

The expression $$2\int (A^2 + B^2 - C^2 + 2AB\cos\omega)^{\frac{1}{2}}\, d\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a)$$

is the value of the difference of corresponding arcs of the two ovals.

The values of ρ for $\omega = 0$, $\omega = \pi$, are

$$\rho = A + B \pm \left\{(A+B)^2 - C^2\right\}^{\frac{1}{2}}, \quad \rho = A - B \pm \left\{(A-B)^2 - C^2\right\}^{\frac{1}{2}};$$

the latter equation giving values of ρ which are to be measured in a direction opposite to that of the values of ρ given by the former equation. That is to say, if A B C D represent the intersections of the ovals with the axis, we have

$$CD = 2\left\{(A+B)^2 - C^2\right\}^{\frac{1}{2}}, \quad AB = 2\left\{(A-B)^2 - C^2\right\}^{\frac{1}{2}},$$

and the integral (a) between the limits 0, π is equivalent to the integral

$$4\int \left\{(A+B)^2 - C^2 - 4AB\sin^2\theta\right\}^{\frac{1}{2}} d\theta,$$

between the limits 0, $\frac{1}{2}\pi$. The difference of the lengths of the two ovals of the Cartesians is therefore equal to the periphery of an ellipse having its principal semi-diameters equal to

$$2\left\{(A+B)^2 - C^2\right\}^{\frac{1}{2}} \text{ and } 2\left\{(A-B)^2 - C^2\right\}^{\frac{1}{2}},$$

i. e., CD and AB.

Again, taking the equation of the Cartesian in the form

$$\rho^2 - 2(A' + B'\cos\omega)\rho - C'^2 = 0;$$

we see that corresponding points lie on non-adjacent parts of the ovals and the integral $$2\int \left\{A'^2 + B'^2 + C'^2 + 2A'B'\cos\omega\right\}^{\frac{1}{2}} d\omega,$$

gives the sum of corresponding arcs.

Moreover, the distance BD is now $2\left\{(A' + B')^2 + C'^2\right\}^{\frac{1}{2}}$, and the distance AC is $2\left\{(A' - B')^2 + C'^2\right\}^{\frac{1}{2}}$, and it follows, as before, that the sum of the lengths of the two ovals is equal to the periphery of an ellipse having its principal semi-diameters equal to BD and AC.

Hence it follows that the length of the outer oval is equal to half the sum of these peripheries, and that of the inner oval is equal to half the difference.

The nodal limaçon is a limiting case. We have (say) $AB = 0$, and the ellipse having CD, AB for principal semi-diameters, is the line 2CD twice over; the other ellipse has AD, AC for principal semi-diameters. The value of loop periphery is therefore as stated in the question.

When the focus-origin is external, corresponding points lie on the same oval. The integral (a) then gives the difference of corresponding arcs. At the same time the corresponding ellipse has its principal semi-diameters equal to AD, BC.

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The limits of integration in this case are 0, and the angle made by the focal tangents with the axis.

The foregoing results suggest that the arc of a Cartesian generally may be expressed by elliptic functions. I have observed that this is actually the case, on account of the special form of the hyper-elliptic integral.

4372. (Proposed by A. L. Pirie, M.A.)—ABC is a triangle inscribed in a circle; from any point O on the circumference perpendiculars are drawn to OA, OB, OC, meeting BC, CA, AB, in D, E, F: prove that D, E, F lie on a straight line which passes through the centre of the circle.

I. *Solution by* E. B. Elliott, M.A.

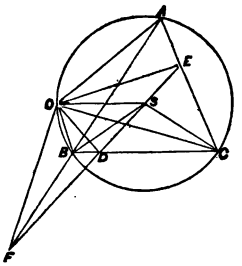
Taking S the centre of the circle, we have

$$\angle DOE = \pi - AOB = ECD,$$

$$\angle EOS = \tfrac{1}{2}\pi - BOS = \tfrac{1}{2}OSB = DCO,$$

$$\angle SOC = OCS;$$

$$\therefore \quad O(DESC) = C(EDOS) = C(DESO).$$



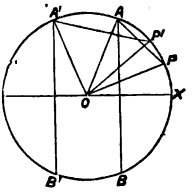
Now these two pencils have a common fourth ray OC, therefore the intersections D, E, S are collinear.

Similarly D, F, S are collinear. Thus D, E, F lie on a straight line passing through S.

II. *Solution by the* Rev. T. J. Sanderson, M.A.

This may be made to depend on the following theorem, which we will first prove.

AB, A′B′ are two equal and parallel chords in a circle; P, P′ are any two points on the circumference. Then the intersections of PA, P′A′; PB, P′B′ lie on a straight line which passes through the centre.



Let $\angle AOX = \theta_1$, $POX = \phi$, $P'OX = \phi'$, $A'OX = \pi - \theta_1$; then the polar equations to AP, A′P′, are

$$r\cos\left\{\theta - \tfrac{1}{2}(\phi + \theta_1)\right\} = a\cos\tfrac{1}{2}(\theta_1 - \phi),$$

$$r\sin\left\{\theta - \tfrac{1}{2}(\phi' - \theta_1)\right\} = a\sin\tfrac{1}{2}(\theta_1 - \phi');$$

and for the intersection of these two lines, we have

$$\frac{\sin\left\{\theta - \tfrac{1}{2}(\phi' - \theta_1)\right\}}{\cos\left\{\theta - \tfrac{1}{2}(\phi - \theta_1)\right\}} = \frac{\sin\tfrac{1}{2}(\theta_1 + \phi')}{\cos\tfrac{1}{2}(\theta_1 - \phi)},$$

or

$$\tan\theta = \tan\tfrac{1}{2}(\phi + \phi') + \frac{\sin\tfrac{1}{2}(\phi' - \phi)}{\cos\tfrac{1}{2}(\phi + \phi')\cos\theta_1};$$

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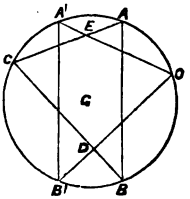
similarly, for the intersection of BP, B′P′, we have

$$\tan\theta = \tan\tfrac{1}{2}(\phi+\phi') + \frac{\sin\tfrac{1}{2}(\phi'-\phi)}{\cos\tfrac{1}{2}(\phi+\phi')\cos(-\theta_1)},$$

which gives the same value of tan θ as before, that is, the two intersections lie on a straight line passing through the centre.

Now, in the Question proposed, OD is perpendicular to OA, and OE to OB; therefore ∠AOE = ∠BOD, and A′B′ is parallel and equal to AB. And by the theorem just proved, O, C being any two points on the circumference, the intersections of OA′, CA; OB′, CB lie on a straight line through the centre: *i.e.*, FGD is a straight line. Similarly, F may be shown to lie on the same straight line.



4239. (Proposed by Professor WOLSTENHOLME, M.A.)—If

$$\frac{\sin(\alpha+\beta-\theta)}{\sin(\alpha+\beta)\cos^2\gamma} = \frac{\sin(\alpha+\gamma-\theta)}{\sin(\alpha+\gamma)\cos^2\beta},$$

β, γ being unequal and $< \frac{1}{2}\pi$, then will each of these be equal to

$$\frac{\sin(\beta+\gamma-\theta)}{\sin(\beta+\gamma)\cos^2\alpha};$$

and

$$\cot\theta = \frac{\sin^2(\alpha+\beta+\gamma)+\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)}{\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)}.$$

Solution by J. J. WALKER, M.A.

From the given equations it readily follows that

$$\cot\theta = \frac{\sin(\gamma+\alpha)\cos(\alpha+\beta)\cos^2\beta - \sin(\alpha+\beta)\cos(\gamma+\alpha)\cos^2\gamma}{\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)\sin(\gamma-\beta)}.$$

Now

$$\begin{aligned}
&4\sin(\gamma+\alpha)\cos(\alpha+\beta)\cos^2\beta \\
&= 2\left\{\sin(\alpha+\beta+\gamma)+\sin(\alpha-\beta+\gamma)\right\}\cos(\alpha+\beta)\cos\beta \\
&= \sin(\alpha+\beta+\gamma)\left\{\cos(\alpha+2\beta)+\cos\alpha\right\} \\
&\qquad + \left\{\sin(\gamma+\alpha)+\sin(\alpha-2\beta+\gamma)\right\}\cos(\alpha+\beta).
\end{aligned}$$

Similarly

$$\begin{aligned}
4\sin(\alpha+\beta)\cos(\gamma+\alpha)\cos^2\gamma &= \sin(\alpha+\beta+\gamma)\left\{\cos(\alpha+2\gamma)+\cos\alpha\right\} \\
&\quad + \left\{\sin(\alpha+\beta)+\sin(\alpha+\beta-2\gamma)\right\}\cos(\gamma+\alpha).
\end{aligned}$$

Hence four times the numerator of cot θ

$$\begin{aligned}
&= \sin(\alpha+\beta+\gamma)\left\{\cos(\alpha+2\beta)-\cos(\alpha+2\gamma)\right\} + \sin(\gamma-\beta) \\
&\quad + \sin(\gamma+\alpha)\cos(\alpha-\beta)\cos 2\beta - \sin(\alpha+\beta)\cos(\gamma+\alpha)\cos 2\gamma \\
&\quad + \cos(\gamma+\alpha)\cos(\alpha+\beta)(\sin 2\gamma - \sin 2\beta) \\
&= 2\left\{\sin^2(\alpha+\beta+\gamma)+\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)\right\}\sin(\gamma-\beta) \\
&\quad + 2\left\{\sin(\gamma+\alpha)\cos(\alpha+\beta)\cos^2\beta - \sin(\alpha+\beta)\cos(\gamma+\alpha)\cos^2\right\}
\end{aligned}$$

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i. e., the numerator of

$$\cot\theta = \left\{\sin^2(\alpha+\beta+\gamma)+\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)\right\}\sin(\gamma-\beta).$$

This value of $\cot\theta$ being symmetrical in α, β, γ would equally follow from

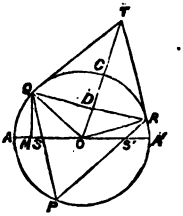
$$\frac{\sin(\alpha+\beta-\theta)}{\sin(\alpha+\beta)\cos^2\gamma} = \frac{\sin(\beta+\gamma-\theta)}{\sin(\beta+\gamma)\cos^2\alpha};$$

and conversely, if the numerator and denominator of $\cot\theta$ were multiplied by $\sin(\gamma-\alpha)$, this last equation might be deduced from it. It does not appear that the equations hold only for values of α, β, γ less than $\frac{1}{2}\pi$.

4371. (Proposed by the Editor.)—Three of the vertices of a quadrilateral move round the curve of an ellipse, two of the sides pass through the foci, and the other two are tangents to the ellipse; show that the locus of the fourth vertex is an ellipse having its centre and axes on those of the given ellipse, its minor semi-axis equal to a, and its major semi-axis equal to $2a^2b^{-1}-b$, where a and b are the semi-axes of the given ellipse.

I. *Solution by the* Rev. T. J. Sanderson, M.A.

Project the ellipse into the circle on the major axis. The ordinates of every point on the curve will thus be multiplied by a factor $\frac{a}{b}$, while their abscissas will remain unaltered.



Let RPQT be the quadrilateral of the question thus projected; and put therein $\angle RPQ = QOT = \phi$, $\angle SOT = \theta$, $OT = r$. Then we have

$$SS'^2 = SP^2+S'P^2-2SP.S'P\cos\phi$$
$$= 2(SO^2+OP^2)-2SP.S'P\cos\phi,$$

or $$a^2(1-e^2) = SP.S'P\cos\phi, \quad (\text{for } OS = ae) \ldots\ldots\ldots\ldots (1).$$

Again $SP.SQ = AS.A'S = AO^2-OS^2$; therefore $SP = \frac{a^2(1-e^2)}{SQ}$;

and $$OQ^2-SQ^2 = ON^2-SN^2 = 2OS.ON-OS^2,$$

therefore $$SQ^2 = a^2(1+e^2)-2a^2e\cos(\theta-\phi);$$

$$\therefore\ SP^2 = \frac{a^2(1-e^2)^2}{1+e^2-2e\cos(\theta-\phi)};\ \text{similarly } S'P^2 = \frac{a^2(1-e^2)^2}{1+e^2+2e\cos(\theta+\phi)}.$$

Substituting in (1), the equation becomes

$$\left\{1+e^2-2e\cos(\theta-\phi)\right\}\left\{1+e^2+2e\cos(\theta+\phi)\right\} = (1-e^2)^2\cos^2\phi,$$

or $$(1+e^2)^2+4e(1+e^2)\sin\theta\sin\phi-4e^2(\cos^2\phi-\sin^2\theta) = (1-e^2)^2\cos^2\phi,$$

or $$(1+e^2)^2-(1-e^2)^2+\sin^2\phi\,(1-2e^2+e^4+4e^2)-4e(1+e^2)\sin\theta\sin\phi$$
$$-4e^2(1-\sin^2\theta) = 0,$$

or $$4e^2\sin^2\theta+(1+e^2)^2\sin^2\phi-4e(1+e^2)\sin\theta\sin\phi = 0;$$

therefore $$2e\sin\theta = (1+e^2)\sin\phi, \text{ and } 4e^2\sin^2\theta = (1+e^2)^2\left(1-\frac{a^2}{r^2}\right);$$

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therefore $r^2 = \dfrac{a^2(1+e^2)^2}{(1+e^2)^2 - 4e^2\sin^2\theta}$, the polar equation of an ellipse.

Its centre and axes lie on the given ellipse, and the major on the minor, and *vice versâ*, which is also geometrically evident.

From the equation or from the figure we see that the length of the major semi-axis is $\dfrac{a(2a^2-b^2)}{b^2}$; therefore the length of the major axis of the required ellipse is

$$\frac{b}{a}\cdot\frac{a(2a^2-b^2)}{b^2} = 2a^2b^{-1}-b,$$

and the minor semi-axis plainly $= a$.

II. *Solution by the* PROPOSER.

Suppose the figure in the foregoing solution to be an ellipse; and let OT meet the curve in C, and QR meet OT in D. Put α, β, γ for the eccentric angles of P, Q, R; and let $\angle AOT = \theta$, $OT = r$, $OD = \rho$. Then, by known properties, we have QD = DR, and

$$r\rho = OC^2 = \frac{b^2}{1-e^2\cos^2\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now the condition that QD = DR, and that PQ, PR pass through the foci, are expressed by the following equations:—

$$a(\cos\beta+\cos\gamma) = 2\rho\cos\theta, \quad b(\sin\beta+\sin\gamma) = 2\rho\sin\theta \quad \ldots\ldots (2),$$

$$\cos\tfrac{1}{2}(\alpha-\gamma) = e\cos\tfrac{1}{2}(\alpha+\gamma), \quad \cos\tfrac{1}{2}(\alpha-\beta) = -e\cos\tfrac{1}{2}(\alpha+\beta) \ldots (3);$$

hence the equation of the locus of T will be obtained by eliminating $\alpha, \beta, \gamma, \rho$ from (1), (2), (3). Eliminating ρ and α, we have

$$r\cos\tfrac{1}{2}(\beta-\gamma) = a\sec\theta\cos\tfrac{1}{2}(\beta+\gamma) = \frac{b}{(1-e^2\cos^2\theta)^{\frac{1}{2}}} \quad \ldots\ldots (4),$$

$$(1+e^2)\sin\tfrac{1}{2}(\beta-\gamma) = 2e\sin\tfrac{1}{2}(\beta+\gamma) \ldots\ldots\ldots\ldots\ldots (5);$$

and eliminating β and γ from (4) and (5), the polar equation of the locus of T is found to be

$$r^2(1-\epsilon^2\sin^2\theta) = a^2, \text{ where } \epsilon^2 = \frac{e^2(e^2+3)}{(1+e^2)^2} \ldots\ldots\ldots\ldots (6),$$

which proves the results stated in the Question.

III. *Solution by* E. B. ELLIOTT, M.A.

Let α, β, γ be the eccentric angles of the three vertices which lie on the ellipse; then the conditions that the chords $a\gamma$ and $\beta\gamma$ pass each through a focus, give

$$e\cos\tfrac{1}{2}(\alpha+\gamma) = \cos\tfrac{1}{2}(\alpha-\gamma), \quad -e\cos\tfrac{1}{2}(\beta+\gamma) = \cos\tfrac{1}{2}(\beta-\gamma).$$

From these, by eliminating γ, we find

$(1+e)^2\tan\tfrac{1}{2}\alpha = (1-e)^2\tan\tfrac{1}{2}\beta$, or $(1+e^2)\sin\tfrac{1}{2}(\alpha-\beta) + 2e\sin\tfrac{1}{2}(\alpha+\beta) = 0$,

which is easily transformed to

$$(1+e^2)^2\cos^2\tfrac{1}{2}(\alpha+\beta) + (1-e^2)^2\sin^2\tfrac{1}{2}(\alpha+\beta) = (1+e^2)^2\cos^2\tfrac{1}{2}(\alpha-\beta).$$

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Hence the fourth vertex, the intersection of tangents at α and β, lies on the curve $$(1+e^2)^2\frac{x^2}{a^2}+(1-e^2)^2\frac{y^2}{b^2}=(1+e^2)^2,$$

that is, on $$\frac{x^2}{a^2}+\frac{b^2y^2}{(2a^2-b^2)^2}=1,$$

which is the ellipse described in the question.

IV. *Solution by* STEPHEN WATSON.

Let e be the focal distance, and denote the points P, Q, R by $x'y'$, x_1y_1, x_2y_2 respectively; then we have

$$(x'+c)\,y_1-y'x_1=cy', \quad (x'-c)\,y_2-y'x_2=-cy' \ \ldots\ldots\ (1, 2),$$

$$a^2y_1^2+b^2x_1^2=a^2b^2=ay_2^2+bx_2^2 \ \ldots\ldots\ (3),$$

and from these, remembering that $a^2y'^2+b^2x'^2=a^2b^2$, we obtain

$$x_1=-\frac{2a^2c+(a^2+c^2)\,x'}{a^2+c^2+2cx'}, \quad y_1=-\frac{b^2y'}{a^2+c^2+2cx'};$$

$$x_2=-\frac{2a^2c-(a^2+c^2)\,x'}{a^2+c^2-2cx'}, \quad y_2=-\frac{b^2y'}{a^2+c^2-2cx'}.$$

Now the equations of DP, EP are

$$b^2x_1x+a^2y_1y=a^2b^2=b^2x_2x+a^2y_2y';$$

hence, substituting the above values of x_1, y_1, x_2, y_2, the results easily give

$$x=-x', \text{ and } a^2yy'=(a^2+c^2)(a^2-x'^2);$$

therefore the equation of the locus required is

$$a^2b^2y^2=(a^2+c^2)^2(a^2-x^2),$$

which represents the ellipse described in the question.

4351. (Proposed by A. W. PANTON, M.A.)—If the roots of the equation $a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}+\ldots=0$ be in arithmetical progression, show that they are obtained from the expression

$$-\frac{a_1}{a_0}\pm\frac{p}{a_0}\left(\frac{3(a_1^2-a_0a_2)}{n+1}\right)^{\frac{1}{2}}$$

by giving p all the values $1, 3, 5, \ldots (n-1)$ when n is even, and all the values $0, 2, 4 \ldots (n-1)$ when n is odd.

I. *Solution by* A. M. NASH.

If n is even, let the roots be

$$\alpha\pm\beta, \quad \alpha\pm3\beta, \ \ldots\ldots, \ \alpha\pm(n-1)\beta \ \ldots\ldots\ (1).$$

If n is odd, let them be

$$\alpha, \quad \alpha\pm2\beta, \quad \alpha\pm4\beta, \ \ldots\ldots, \ \alpha\pm(n-1)\beta \ \ldots\ldots\ (2).$$

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In either case sum of roots $= n\alpha = -\frac{na_1}{a_0}$, therefore $\alpha = -\frac{a_1}{a_0}$.

Sum of squares of roots $= \frac{n^2a_1^2}{a_0^2} - n(n-1)\frac{a_2}{a_0}$

$= (\alpha \pm \beta)^2 + (\alpha \pm 3\beta)^2$, or $\alpha^2 + (\alpha \pm 2\beta)^2 + \ldots\ldots$

$= n\alpha^2 + 2\beta^2 \{1^2 + 3^2 + \ldots + (n-1)^2\}$, or $n\alpha^2 + 2\beta^2 \{2^2 + 4^2 + \ldots + (n-1)^2\}$

$= n\alpha^2 + \frac{1}{3}(n-1)n(n+1)\beta^2$ in either case;

therefore $\frac{n^2a_1^2}{a_0^2} - n(n-1)\frac{a_2}{a_0} = \frac{na_1^2}{a_0^2} + \beta^2\frac{(n-1)n(n+1)}{3}$;

therefore $\beta = \frac{1}{a_0}\left\{\frac{3(a_1^2 - a_0a_2)}{n+1}\right\}^{\frac{1}{2}}$;

hence the roots may be obtained from the expression given in the question.

II. *Solution by* W. J. Greenfield.

Let S_1, S_2 denote the sums of the first and second powers of the roots, and let d be the common difference of the roots.

Then it may be shown that

$$S_2 = \frac{1}{n} \cdot S_1^2 + \frac{n(n-1)(n+1)}{3 \cdot 4} \cdot d^2;$$

therefore $$S_1^2 - S_2 = \frac{n-1}{n} \cdot S_1^2 - \frac{n(n-1)(n+1)}{3 \cdot 4} \cdot d^2.$$

But $$S_1 = -n\frac{a_1}{a_0}, \text{ and } S_1^2 - S_2 = n(n-1)\frac{a_2}{a_0}.$$

Hence $$n(n-1)\frac{a_2}{a_0} = n(n-1) \cdot \frac{a_1^2}{a_0^2} - \frac{n(n-1)(n+1)}{3 \cdot 4} \cdot d^2;$$

therefore $a_0a_2 = a_1^2 - \frac{n+1}{3 \cdot 4} \cdot a_0^2 \cdot d^2$; therefore $\frac{d^2}{4} = \frac{(3a_1^2 - a_0a_2)}{(n+1)a_0^2}$.

Again, it may be shown that the rth term of the series of roots

$a_1, a_1 + d, \ldots$ &c., is $$\frac{1}{n}S_1 + \frac{2r-n-1}{2} \cdot d.$$

Therefore, the general expression for the roots becomes

$$-\frac{a_1}{a_0} + \frac{2r-n-1}{a_0} \cdot \left\{\frac{3(a_1^2 - a_0a_2)}{n+1}\right\}^{\frac{1}{2}}.$$

Now $2r-n-1$ is odd if n is even, and even if n is odd, and its value varies from $-(n-1)$ to $+(n-1)$. If, then p have the values 1, 3, 5, &c. up to $(n-1)$ when n is even; or 0, 2, 4, &c., up to $(n-1)$, when n is odd, the roots will be given by the formula in the question.

4311. (Proposed by T. T. Wilkinson, F.R.A.S.)—Find the point in the side of a given rectangular billiard table, adjacent to the pocket, from

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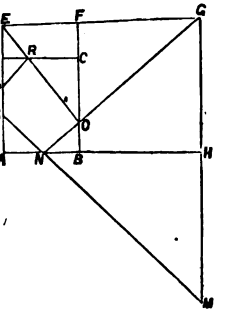
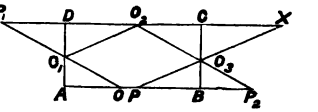
which if a ball be struck so as to hit each of the other three sides in succession, and enter the pocket, it may then have run a length equal to the perimeter of the table.



I. *Solution by* HUGH MURPHY.

Let P be the pocket, S the point required, and SNORP the path of the ball after being struck. Then if E be the reflexion of P, G of E, and M of G, the point M is fixed, and MS is equal to the length of path, that is, to the perimeter of the table. Hence we have only to inflect a line from M to AE equal to a given line, and the point S becomes known.

II. *Solution by the* PROPOSER.



Let ABCD be the table; P the pocket, and O the point required. Then, if O_1, O_2, O_3 be the points of incidence on AD, DC, CB, respectively, we have the angle $OO_1A = DO_1O_2 = DO_1P_1$, when OO_1 is produced to meet CD in P_1; whence $O_1P_1 = O_1O_2$; and $OP_1 = OO_1 + O_1O_2$. Produce O_2O_3 to meet AB produced in P_2, and similarly, $O_3P_2 = PO_3 + O_3O_2$. But, by construction, $OP_1O_2P_2$ is a parallelogram, and we have therefore to make $OP_1 = O_2P_2 = AB + BC = CD + DA$. Hence the following construction is obvious:—With centre P and radius $= AB + BC$ describe a circle cutting DC produced in X; join PX cutting BC in O_3; make the angle $CO_3O_2 = PO_3B$; draw O_2O_1, O_1O respectively parallel to PO_3, O_3O_2; and the point O is that required. The demonstration is obvious.

4366. (Proposed by Professor CROFTON, F.R.S.)—Show that a normal stress, acting all round a closed convex curve, of an intensity everywhere inversely proportional to the radius of curvature, makes equilibrium. Find the resultant of the stress on any arc of the curve.

I. *Solution by* Professor TOWNSEND, F.R.S.

The stress in question being what would be produced by the uniform tension of a flexible cord, strained without friction round the curve, and held in equilibrium without external force by its reaction, its resultant, for the entire length of the curve, is consequently evanescent, and, for any arc of it, passes through the intersection of the two terminal tangents, bisects the internal angle between them, and is equal to twice the product of the uniform tension into the cosine of the semi aforesaid angle bisected by it.

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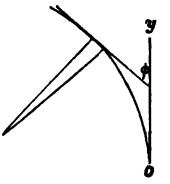
II. *Solution by the* Rev. T. J. Sanderson, M.A.



The stress at any point $= \frac{k}{\rho}$ suppose. The stress on an element δs at this point, resolved parallel to a fixed straight line $Oy = \frac{k}{\rho}\,\delta s \sin\phi$.

Therefore stress on whole curve resolved parallel to Oy

$$= k\int_0^{2\pi}\frac{1}{\rho}\sin\phi\,ds = k\int_0^{2\pi}\frac{d\phi}{ds}\sin\phi\,ds$$

$$= k\int_0^{2\pi}\sin\phi\,d\phi = 0.$$

And Oy may be taken in any direction, therefore there is equilibrium.

Take any arc measured from O, with an intrinsic angle ϕ'.
The resolved stress on this arc parallel to Oy

$$= k\int_0^{\phi'}\sin\phi\,d\phi = k\,(1-\cos\phi')\,;$$

the resolved stress perpendicular to $Oy = k\int_0^{\phi'}\cos\phi\,d\phi - \sin\phi'$,

therefore

resultant stress on this arc $= \left\{(1-\cos\phi')^2 + \sin^2\phi'\right\}^{\frac{1}{2}} = 2\sin\tfrac{1}{2}\phi'$.

3752. (Proposed by Asher B. Evans, M.A.)—Divide unity into three positive parts such that the sum of their squares, and also the sum of their cubes, shall be square numbers.

I. *Solution by* Dr. Hart; S. Bills; *and others.*

Let $\frac{x}{x+y+z}$, $\frac{y}{x+y+z}$, $\frac{z}{x+y+z}$ be the parts; then their sum is 1; the sum of their squares is

$$\frac{x^2+y^2+z^2}{(x+y+z)^2} = \square, \text{ or } x^2+y^2+z^2 = \square\,;$$

and the sum of their cubes is

$$\frac{x^3+y^3+z^2}{(x+y+z)^3} = \square, \text{ or } \frac{x^3+y^3+z^3}{x+y+z} = \square\,;$$

therefore we are to make

$$x+y+z = \square,\quad x^2+y^2+z^2 = \square,\quad \text{and}\quad x^3+y^3+z^3 = \square,$$

which has been done in the solution of Question 3496, on p. 30 of

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Vol. XVII. of the *Reprint*, where the numbers found are

$$x = 412095790665, \quad y = 273267248670, \quad z = 194479084666.$$

Hence the numbers or parts of unity are

$$\frac{412095790665}{879842124001}, \quad \frac{273267248670}{879842124001}, \quad \frac{194479084666}{879842124001}.$$

II. *Solution by the* PROPOSER.

Let $m(r^2-1)x$, $m(r^2-1)y$, and $2mr(x+y)$ be the three required parts; then we must have

$$m(r^2-1+2r)(x+y) = 1, \quad m^2[(r^2+1)^2(x^2+y^2)+8r^2xy] = \square \ldots (1, 2),$$
$$m^3[(r^2-1)^3(x^3+y^3)+8r^3(x+y)^3] = \square \ldots\ldots\ldots\ldots (3).$$

By dividing (3) by (1), we obtain for the quotient

$$m^2[(r^2-1)^2-2r(r^2-1)+4r^2](x^2-xy+y^2) + \frac{16m^2r^3xy}{r^2-1+2r};$$

and this will be simplified by making

$$(r^2-1)^2-2r(r^2-1)+4r^2 = \square = (r^2-r+1)^2 \text{ suppose};$$

for then $r=4$, and the required conditions become

$$23m(x+y) = 1, \quad x^2+axy+y^2 = \square, \quad \text{and} \quad x^2-bxy+y^2 = \square \ldots (4, 5, 6);$$

where $$a = \frac{128}{289} \quad \text{and} \quad b = \frac{2351}{3887}.$$

Conditions (5) and (6) are satisfied (*Reprint*, Vol. XVIII. p. 31) by taking $x = (4+a+b)(4-a-b)$, and $y = 8(b-a)$. From (4), $m = \frac{1}{23(x+y)}$; whence we find the three required parts to be

$$\frac{15}{23}\left(\frac{x}{x+y}\right) = \frac{282076826696385}{469883729891593}, \quad \frac{15}{23}\left(\frac{y}{x+y}\right) = \frac{24369084102480}{469883729891593},$$
$$\frac{8}{23}\left(\frac{x+y}{x+y}\right) = \frac{163437819092728}{469883729891593}.$$

4215. (Proposed by S. WATSON.)—Let ABC be a triangle; O its centroid; D_1, E_1, F_1 the middle points of BC, CA, AB; D_2, E_2, F_2 the middle points of E_1F_1, F_1D_1, D_1E_1; and so on. Then if P be any point in the plane of the triangle, prove that

$$PD_n^2 + PE_n^2 + PF_n^2 = 3PO^2 + \frac{1}{3.4^n}(AB^2+BC^2+CA^2).$$

Solution by R. F. SCOTT.

Taking the point P as origin, let α, β, γ be the vectors to A, B, C; then, if u_n, v_n, w_n denote the vectors PD_n, PE_n, PF_n, we have

$$2u_n = v_{n-1} + w_{n-1}, \text{ with two similar equations.}$$

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Hence $2u_n = u_{n-2} + u_{n-1}$; therefore $u_n = \sigma + \left(\frac{-1}{2}\right)^n 2\left(\sigma - \frac{\beta+\gamma}{2}\right)$,

where σ is the vector PO. We get similar expressions for v_n and w_n.

Hence adding their squares we have

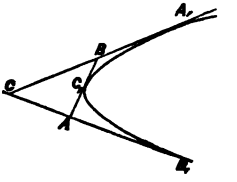
$$PD_n^2 + PE_n^2 + PF_n^2 = 3PO^2 + \frac{4}{4^n}(DO^2 + EO^2 + FO^2)$$

$$= 3PO^2 + \frac{1}{3.4^n}(AB^2 + BC^2 + CA^2).$$

4389. (Proposed by the Rev. Dr. Booth, F.R.S.)—A triangle circumscribes a parabola; prove that (1) the square of the product of the three sides of the triangle is equal to the product of the six tangents drawn to the curve from the three angles of the triangle; and (2) the product of the three focal vectors drawn to the angles of the triangle is equal to the product of the three focal vectors drawn to the points of contact.

Solution by the Proposer.

1. In Booth's *New Geometrical Methods*, pp. 43, 47, it is shown that if a parabola be referred to a pair of tangents as axes of coordinates, its tangential equation is $g\xi\upsilon + h\xi + h_1\upsilon = 0 \equiv F$ ..... (a).



Let CAB be the circumscribing triangle, $CA = a$, $CB = b$; then as CA and CB are the tangential coordinates of the tangent AB to the parabola, we may put a for ξ^{-1}, and b for υ^{-1}, in the equation (a), which thus becomes

$$g + hb + h_1 a = 0 \quad \ldots\ldots\ldots (b).$$

We may obtain the value of the tangent CA_1 by putting $\xi^{-1} = 0$ in the equation (a); whence $\frac{1}{\upsilon}$, or $CA_1 = -\frac{g}{h}$, and consequently

$$CA_1 - CB = BA_1 = -\left(\frac{g}{h} + b\right).$$

In like manner, $AB_1 = -\left(\frac{g}{h_1} + a\right)$, consequently

$$BA_1 . AB_1 = \left(\frac{g}{h} + b\right)\left(\frac{g}{h_1} + a\right) = \frac{g}{hh_1}(g + h_1 a + hb) + ab.$$

But as $g + h_1 a + hb = 0$, by (b), we have $BA_1 . AB_1 = ab$.

Taking the origin of coordinates successively at the other two angles of the triangle, and multiplying the results, we shall finally obtain

$$BA_1 . AB_1 . CB_1 . BC_1 . CA_1 . AC_1 = (abc)^2.$$

2. A somewhat similar proof may be given of the second theorem. We have also $\frac{CB}{A_1B} = \frac{AC_1}{C_1B} = \frac{AB_1}{AC}$, as each ratio $= \frac{-bh}{g+bh}$.

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4319. (Proposed by B. WILLIAMSON, M.A.)—Give an elementary proof of Landen's theorem, that the length of any arc of an hyperbola can be expressed by means of the arcs of two ellipses.

Solution by the PROPOSER.

In any triangle we have $c = a\cos B + b\cos A$.

Now if the external angle opposite the side c be denoted by C, we have

$$cdC = (a\cos B + b\cos A)(dA + dB).$$

Hence, regarding a and b as constants, and the rest as variable, we have

$$\int cdC = \int a\cos B dA + \int b\cos A dB + 2a\sin B + \text{constant},$$

or
$$\int_0^{A_1+B_1} (a^2+b^2+2ab\cos C)^{\frac{1}{2}}\, dC = \int_0^{A_1} (a^2-b^2\sin^2 A)^{\frac{1}{2}}\, dA + \int_0^{B_1} (b^2-a^2\sin^2 B)^{\frac{1}{2}}\, dB + 2a\sin B_1,$$

where A_1 and B_1 are any angles connected by the relation

$$a\sin B_1 = b\sin A_1.$$

Now, let $a > b$, and the integral at the first side of the equation represents an arc of the ellipse, whose semi-axes are $a+b$ and $a-b$; also the integral $\int_0^{A_1} (a^2-b^2\sin^2 A)^{\frac{1}{2}}\, dA$ represents an arc of the ellipse of semi-axis major a, and eccentricity $\frac{b}{a}$; while $\int_0^{B_1} (b^2-a^2\sin^2 B)^{\frac{1}{2}}\, dB$ represents the difference between an hyperbolic arc and a certain right line; by well known elementary theorems. Hence Landen's Theorem follows immediately.

3691. (Proposed by Dr. HART.)—In a plane triangle, it is required to find integers for the sides, diameter of the inscribed circle, and the side of the inscribed square, such that the sum of the diameter and side shall be a square.

Solution by ASHER B. EVANS, M.A.

Let a, b, c, d, s represent the three sides, the diameter of the inscribed circle, and the side of the inscribed square resting upon the side c. Then we must have

$$d + s = \square \quad \ldots\ldots\ldots\

and $$s = \frac{abc}{ab+c^2} = \frac{2mn\,(m^2-n^2)(m^2+n^2)\,xy}{2mn\,(m^2-n^2)+(m^2+n^2)^2} \quad \ldots\ldots\ldots\ldots (3).$$

Let $$x = 2mn\,(m^2-n^2)+(m^2+n^2)^2;$$

then, from (2) and (3), we have

$$d = 2n\,(m-n)\left\{2mn\,(m^2-n^2)+(m^2+n^2)^2\right\} y \quad \ldots\ldots\ldots\ldots (4).$$

$$s = 2mn\,(m^2-n^2)(m^2+n^2)\,y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

It is evident from (4) and (5) that (1) will be satisfied if

$$y = 2n\,(m-n)\left\{2mn\,(m^2-n^2)+(m^2+n^2)^2\right\} + 2mn\,(m^4-n^4).$$

Let $m = 2$ and $n = 1$, $x = 37$, $y = 134$, $a = 14974$, $b = 19832$, $c = 24790$, $d = 9916$, $s = 8040$, $\sqrt{d+s} = 134$.

4055. (Proposed by Dr. HART.)—Find any number of cube numbers whose sum is a cube.

Solution by ARTEMAS MARTIN.

Let rm, $(q-rm)$, sm, p_1q, p_2q, p_3q ..... $p_{n-2}q$ be the roots of n cubes, and $sm + \frac{r^2q}{s^2}$ the root of their sum. Then we must have

$$r^3m^3+(q-rm)^3+s^3m^3+p_1^3q^3+p_2^3q^3+p_3^3q^3+\ldots\ldots+p_{n-2}^3q^3 = \left(sm+\frac{r^2q}{s^2}\right)^3.$$

Expanding and reducing, we get

$$\frac{m}{q} = \frac{s^6-r^6+s^6\,(p_1^3+p_2^3+p_3^3+\ldots\ldots+p_{n-3}^3)}{3rs^3\,(s^3+r^3)}.$$

When $n = 3$, we have $$\frac{m}{q} = \frac{s^6-r^6}{3rs^3\,(s^3+r^3)} = \frac{s^3-r^3}{3rs^3};$$

hence the roots of three numbers are $r\,(s^3-r^3)$, $r\,(r^3+2s^3)$, $s\,(s^3-r^3)$, and the root of their sum is $s\,(2r^3+s^3)$.

If $r=1$ and $s=2$, we have $7^3+17^3+14^3 = 20^3$.

When $n=4$, we have $$\frac{m}{q} = \frac{s^6-r^6+p_1^3s^6}{3rs^3\,(s^3+r^3)}.$$

Take $s=1$, $r=2$, $p_1=4$, then $\frac{m}{q} = \frac{1}{54}$, and $1^3+2^3+52^3+216^3 = 217^3$.

If $n=5$, we have $$\frac{m}{q} = \frac{s^6-r^6+p_1^3s^6+p_2^3s^6}{3rs^3\,(s^3+r^3)}.$$

Take $$s=1,\ r=2,\ p_1=2,\ p_2=4;$$

then we have $\frac{m}{q} = \frac{1}{6}$, and $1^3+2^3+4^3+12^3+24^3 = 25^3$.

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3577. (Proposed by S. WATSON.)—From the extremity A of the major axis of a given ellipse a chord AP is drawn, and another chord PQ perpendicular to AP. Find the equation and area of the locus of the middle point of PQ.

Solution by the PROPOSER.

Denoting the point P by (x', y'), the equation of PQ is

$$y - y' = -\frac{a + x'}{y'}(x - x') \quad \ldots\ldots (1),$$

and of the diameter conjugate to that parallel to PQ,

$$y = \frac{b^2 y'}{a^2 (a + x')}\, x \quad \ldots\ldots (2),$$

and this bisects PQ. Eliminate x' from (1) and (2); then we have

$$y' = \frac{a^2 y (a^2 y^2 + b^2 x^2 + a b^2 x)}{a^4 y^2 + b^4 x^2} \quad \ldots\ldots (3).$$

Again, from (2), $x' = \frac{b^2 x}{a^2 y} y' - a$, and this put in $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$, gives

$$(a^6 y^2 + b^6 x^2)\, y' = 2 a^3 b^4 x y \quad \ldots\ldots (4).$$

Hence the required equation of the locus is

$$\frac{a^2 y^2 + b^2 x^2 + a b^2 x}{a^4 y^2 + b^4 x^2} = \frac{2 a b^4 x}{a^6 y^2 + b^6 x^2}$$

or
$$r = \frac{a b^2 \cos\theta \left\{ b^6 \cos^2\theta - a^4 (a^2 - 2b^2) \sin^2\theta \right\}}{(a^6 \sin^2\theta + b^6 \cos^2\theta)(a^2 \sin^2\theta + b^2 \cos^2\theta)}.$$

Half the area lies between θ from 0 to $\frac{1}{2}\pi$; hence the whole area is

$$\int_0^{\frac{1}{2}\pi} r^2 d\theta = a^2 b^4 \int_0^{\infty} \frac{\left\{ b^6 - a^4 (a^2 - 2b^2) z^2 \right\}^2 dz}{(a^6 z^2 + b^6)^2 (a^2 z^2 + b^2)^2} \quad (\text{where } z = \tan\theta)$$

$$= \frac{a^2 b^4}{(a^2 + b^2)^2} \int_0^{\infty} \left(\frac{2 a^2 b^4}{a^6 z^2 + b^6} - \frac{a^2 - b^2}{a^2 z^2 + b^2} \right)^2 dz$$

$$= \frac{a^2 b^4}{(a^2 + b^2)^2} \int_0^{\infty} dz \left\{ \frac{4 a^4 b^8}{(a^6 z^2 + b^6)^2} + \frac{(a^2 - b^2)^2}{(a^2 z^2 + b^2)^2} - \frac{4 a^2 b^2}{a^2 + b^2} \left(\frac{a^4}{a^6 z^2 + b^2} - \frac{1}{a^2 z^2 + b^2} \right) \right\}$$

$$= \frac{a^6 - 5 a^4 b^2 + 11 a^2 b^4 + b^6}{4 (a^2 + b^2)^3}\, a b \pi.$$

When $a = b$, this becomes $= \frac{1}{4} a^2 \pi$, as it ought.

4332. (Proposed by H. S. MONCK.)—The form of the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ implies that if on the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$,

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perpendiculars be drawn from any point of the circumference, their product will be constant. The equation of the ellipse seems to imply a corresponding property with regard to the curve and the imaginary asymptotes, viz., $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$. Draw a pair of lines which shall possess the property in question.

Solution by the PROPOSER.

Through the minor axis of the ellipse draw a plane, making with the plane of the ellipse an angle whose cosine is $+a^{-1}b$, and to this plane draw a perpendicular through the centre of the ellipse. Then if, on this last drawn line, a perpendicular be drawn from any point on the elliptic circumference, its value is constant, and $= b$. For the ellipse projects orthogonally on the plane in question into a circle whose radius is b, and the perpendicular from any point of the elliptic circumference to the last-drawn line will form one side of a rectangle, whose opposite side is a radius of this circle and therefore $= b$.

A second line is found by making the cosine of the angle between the planes $= -a^{-1}b$. The length of the perpendicular here also is $= b$, and their product is $= b^2$.

If the ellipse becomes a circle, the two lines so drawn will coincide. Indeed, since $\sqrt{(-1)}$ has two equal and opposite values $x + \sqrt{(-1)}\,y = 0$ and $x - \sqrt{(-1)}\,y = 0$ must represent the same line.

4375. (Proposed by J. J. WALKER, M.A.)—If α, β, γ are the angles which the bisectors of the sides of a plane triangle drawn from A, B, C respectively make with the side BC, prove that

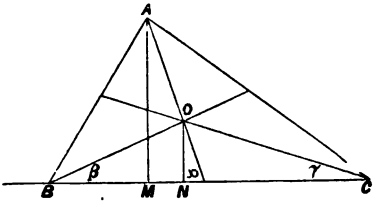
$$\tan\alpha\,(\tan\gamma\tan B - \tan\beta\tan C) - (\tan\beta + \tan\gamma)\tan B\tan C + \tan\beta\tan\gamma\,(\tan B + \tan C) = 0.$$

Solution by the Rev. T. J. SANDERSON, M.A.

$$\text{Tan B} + \tan C = \frac{AM}{BM} + \frac{AM}{CM} = \frac{a\,.\,AM}{BM\,.\,CM},$$

$$\tan B\tan C = \frac{AM^2}{BM\,.\,CM};$$

$$\tan\beta + \tan\gamma = \frac{ON}{BN} + \frac{ON}{CN} = \frac{a\,.\,ON}{BN\,.\,CN},$$



$$\tan\beta\tan\gamma = \frac{ON^2}{BN\,.\,CN};$$

$$\tan\gamma\tan B - \tan\beta\tan C = \frac{ON\,.\,AM}{CN\,.\,BM} - \frac{ON\,.\,AM}{BN\,.\,CM} = ON\,.\,AM\,.\,\frac{BN\,.\,CM - BM\,.\,CN}{BN\,.\,BM\,.\,CN\,.\,CM};$$

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therefore $$\frac{(\tan B + \tan C)\tan\beta\tan\gamma(\tan\beta+\tan\gamma)\tan B\tan C}{\tan\gamma\tan\beta-\tan B\tan C}$$

$$= \frac{a.AM.ON^2 - a.AM^2.ON}{ON.AM(BN.CM-BM.CN)} = a.\frac{ON-AM}{BN.CM-CN.BM}$$

$$= a.\frac{ON-AM}{a.MN} = -\tan\alpha$$

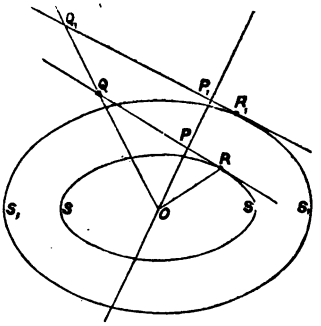
[since $BN.CM-CN.BM = (a-CN)(CN+MN)-CN(a-CN-MN)$
$= a.CM + a.MN - CN^2 - CN.MN - a.CN + CN^2 + CN.MN = a.MN$];
therefore $\tan\alpha(\tan\gamma\tan B - \tan\beta\tan C) - (\tan\beta+\tan\gamma)\tan B\tan C$
$+ \tan\beta\tan\gamma(\tan B + \tan C) = 0.$

4370. (Proposed by Professor TOWNSEND, M.A.)—A rigid body, in unconstrained equilibrium in free space, being supposed set in motion by a single impulsive force, show directly that

a.) If the central axis of initial rotation be a principal axis of the body, the initial motion is one of pure rotation; and, conversely, if the initial motion be one of pure rotation, the central axis of initial rotation is a principal axis of the body.

(*b*.) For different initial axes of pure rotation having a common direction in the body, and lying consequently in a plane passing through its centre of gravity, the locus of the corresponding centres of percussion is a right line passing through the centre of gravity and lying in the plane of the axes.

Solution by the PROPOSER.



Let F be the force, O the centre of gravity of the body, OR the radius of its central ellipsoid of gyration S perpendicular to the plane of OF, OP the perpendicular through O to the plane touching S at R, and OQ the perpendicular through O to the line OR in the plane POR; then since, on well known principles, the initial motions, linear and angular, at O are parallel to F and round OP respectively; hence, if the central axis of initial rotation, necessarily in all cases parallel to OP, be principal, and therefore in the plane POR, the direction of F, necessarily in all cases perpendicular to OR, must be perpendicular also to the plane POR, and therefore to OP; and, conversely, if the direction of F, necessarily in all cases perpendicular to OR, be perpendicular also to OP, and therefore to the plane POR, the central axis of initial rotation, necessarily in all cases parallel to OP, must lie in the plane POR; and therefore, &c., as regards (*a*).

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Let R_1 be the point at which any axis of initial pure rotation, parallel to the line OP and lying in the plane POR, is intersected at right angles by a quadric S_1 of the system confocal with S, P_1 the intersection with OP of the plane touching S_1 at R_1, and Q_1 the corresponding centre of percussion of the force F_1 originating the rotation, which, as is well known, lies on the line P_1R_1, and at the side of OP_1 opposite to that of R_1; then, the two triads of points $P_1Q_1R_1$ and PQR being, as is well known, complanar with each other and with O, since, on known principles, $P_1Q_1 . P_1R_1 = OP^2 = PQ . PR$, and $OP_1 . P_1R_1 = OP . PR$, therefore $P_1Q_1 : OP_1 = PQ : OP$; and therefore, &c., as regards (*b*).

N.B.—If the body were supposed set in motion, not as above by the action of a single impulsive force, but by the simultaneous action of a number of such forces instead; then, unless the direction of the force resulting from their transference to the centre of gravity of the body were perpendicular to the axis of the moment resulting from the same transference, in which case the entire system of forces would, as is well known, be equivalent to a single force, the above properties would of course cease to be true.

4406. (Proposed by J. L. McKenzie.)—Find the least triangle whose sides are integers in arithmetical progression, and the perpendiculars on the sides from the opposite angles integers in harmonical progression.

Solution by H. S. Monck.

In order that the area of the triangle should be integral, the three sides must be of the form $2a+b$, $2a$, and $2a-b$ (b being less than a), and then the area will be $a\left\{3(a^2-b^2)\right\}^{\frac{1}{2}}$; and in order that the area may be integral, it is evident that $3(a^2-b^2)$ must be a perfect square. Instead of seeking directly for the first triangle whose perpendiculars are integral, I will seek for the first triangle in which the area is divisible by each of the three sides; then it is evident that the sides of this triangle will be double those of the one required. Now if $3(a^2-b^2) = d^2$ it is evident that d^2 is divisible by 9. It is also divisible by 4, since the whole expression for the area must be divisible by $2a$. Hence d^2 is of the form $36m^2$; and making $m = 1, 4, 9, 16, 25$, in succession, we can see if any of the values so obtained will make the area divisible by $2a+b$ and $2a-b$ respectively. If $m=1$, we have $(a+b)(a-b) = 12$, and the only integral solution of this (for an integral solution requires that $a+b$ and $a-b$ should both be whole numbers) is $a+b=6$ and $a-b=2$, which does not suit. Examining the integral solutions for $m^2=4$, $m^2=9$, $m^2=16$, we find that none of them (and they are not numerous) give an area divisible by both $2a+b$ and $2a-b$. With $m^2=25$ we get for the first time such a solution, viz. $(a+b)(a-b) = 300$, $a+b=30$, $a-b=10$. The sides of this triangle are 30, 40, 50; and the corresponding sides of the triangle in the Question are 15, 20, 25. It is evident that no higher value of m^2 would give a lower solution than this: the lowest solution of $(a+b)(a-b) = 396$ (which results from making $m^2=36$) giving $2a=42$ instead of 40.

The defect of the solution 15, 20, 25 is that the triangle is a right-angled triangle, and two of the perpendiculars are therefore identical with

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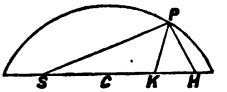
two of the sides. I believe the lowest triangle in which the sides and perpendiculars are really distinct is that whose sides are 845, 910, 975.

Generally, if $3(a^2-b^2)$ is a perfect square, a solution will be given by the triangle whose sides are $(4a^2-b^2)(2a-b)$, $(4a^2-b^2)2a$ and $(4a^2-b^2)(2a+b)$. If $(4a^2-b^2)$ is divisible by 3, there will be a lower solution.

4206. (Proposed by R. TUCKER, M.A.)—P is a point on an ellipse (foci S, H), and K is a point on the major axis such that PK = PS or PH; show (1) that PK touches a sextic; also (2) find the intersection of PK with the similar line for the point D (CD being conjugate to CP).

Solution by the PROPOSER.



The equation to PK is

$$ay(e-\cos\phi) = b\sin\phi(x-2a\cos\phi+ae), \text{ or}$$
$$ab\sin 2\phi - b(x+ae)\sin\phi - ay\cos\phi + aey = 0 \ldots(1).$$

Hence (Salmon's *Higher Plane Curves*, 2nd ed., p. 67), the envelope required is

$$\left\{12a^2b^2 - 3\left[a^2y^2+b^2(x+ae)^2\right]+4a^2e^2y^2\right\}^3$$
$$= \left\{72a^3b^2ey + 9\left[a^2y^2+b^2(x+ae)^2\right]aey + 54\,a^2b^2y(x+ae) - 8a^3e^3y^3\right\}^2.$$

The equation to the conjugate is

$$ab\sin 2\phi + b(x+ae)\cos\phi - ay\sin\phi - aey = 0 \quad \ldots\ldots\ldots\ldots (2),$$

or, putting the equations into the form

$$ab\sin 2\phi - aey = ay\sin\phi - b(x+ae)\cos\phi,$$
$$ab\sin 2\phi + aey = ay\cos\phi + b(x+ae)\sin\phi,$$

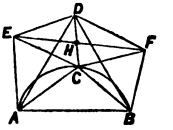
we get, squaring, and then taking sum and difference of these results

$$2(a^2b^2\sin^2 2\phi + a^2e^2y^2) = b^2(x+ae)^2 + a^2y^2, \quad \tan 2\phi = \frac{b^2(x+ae)^2 - a^2y^2}{2aby(x-ae)};$$

whence, eliminating ϕ, we get readily the required equation.

4409. (Proposed by GEOMETRICUS.)—The base and vertical angle of a triangle are given, and equilateral triangles are constructed on the two sides; find the locus of the middle point of the line joining the vertices of the equilateral triangles.

I. *Solution by* CHRISTINE LADD; H. MURPHY; *and others.*



Upon the base AB of the triangle ABC construct an equilateral triangle ADB, and draw the lines as in the figure. Then the triangles EAD and ACB give ED = CB = CF; and, in like manner, FD = EC; therefore EDFC is a parallelogram, and the diagonal DC is bisected in H; hence, as the point D and the segment ACB are given, the locus of H is a circle that can easily be constructed.

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II. *Solution by* R. TUCKER, M.A.

Let the equilateral triangles be ACE, BCF, and H the middle point of DE [see Figure to foregoing solution]; then, taking AB and its perpendicular bisector for axes, and putting (x, y) for the coordinates of H, and (h, k) for those of C, we have

$$2y = b\sin(60° + C) + c\sin(60° + B) = \tfrac{1}{2}a\sqrt{3} + k \ldots\ldots\ldots\ldots (1);$$

also $a^2 + BE^2 = 2BH^2 + 2FH^2$, $b^2 + AF^2 = 2EH^2 + 2AH^2$, and $AF = BE$,

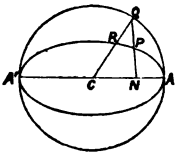
therefore $2(BH^2 - AH^2) = a^2 - b^2$, whence $2x = h$ ..... (2).

Now C lies on a circle whose equation is $h^2 + (k-\lambda)^2 = r^2$, where λ is the ordinate of the centre, and r the radius; hence by (1) and (2) we get, for the locus of H, $4(x^2+y^2) - 4y(\frac{1}{2}a\sqrt{3} + \lambda) = r^2 - (\frac{1}{2}a\sqrt{3} + \lambda)^2$,

which is the equation of a circle.

4345. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—Prove that the semi-diameter of an ellipse along the excentric radius of the circle drawn on the major axis, is equal to the central perpendicular drawn on the tangent to the ellipse through the point where the ordinate meets the ellipse.

Solution by J. L. MCKENZIE.



Let ϕ = the excentric angle QCN; then

$$CN = a\cos\phi, \quad NP = b\sin\phi,$$

and the equation of the tangent at P being

$$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1,$$

the length of the perpendicular on the tangent from the origin is $p = \left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)^{-\frac{1}{2}}$.

But writing $CR \cdot \cos\phi$ for x, and $CR \cdot \sin\phi$ for y, in the equation of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{1}{CR^2} = \frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}$; therefore $CR = p$.

4127. (Proposed by T. T. WILKINSON, F.R.A.S.)—If P be the intersection, and L, M, N the feet of the perpendiculars opposite to A, B, C respectively; prove that

$$AP^2 \cdot BP^2 \cdot CP^2 = 8R^3 \cdot LP \cdot MP \cdot NP.$$

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Solution by PROPOSER; A. B. EVANS, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

$2\triangle BPC = BP \cdot CP \cdot \sin BPC = BP \cdot CP \sin CPM = BP \cdot CP \sin A$ ... (1),

$2\triangle BPC = BC \cdot PL = 2R \cdot PL \cdot \sin A$ ..... (2).

From (1) and (2), we have $BP \cdot CP = 2R \cdot PL$ ..... (3).

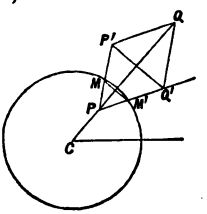
Similarly $CP \cdot AP = 2R \cdot PM$, and $AP \cdot BP = 2R \cdot PN$ ..... (4, 5).

By multiplying together (3), (4), (5), we readily obtain the required expression.

4320. (Proposed by J. J. SYLVESTER, F.R.S.)—If two points, taken equi-distant from one of the angles of a jointed equilateral rhomb on the adjacent sides, move in the same circle, and the opposite angle in a circle passing through the centre of the first, prove that the angle first named will move in a certain third circle.

I. *Solution by* Professor WOLSTENHOLME, M.A.

Let PP'QQ' be the rhomb; M, M' the points moving on a fixed circle with centre C; CPQ a straight line perpendicular to MM' and P'Q'; then, putting $PM = PM' = c$, $CM = CM' = b$, a = a side of the rhomb, 2ϕ the angle P of the rhomb, we have



$$CQ = CP + 2a\cos\phi \equiv r + 2a\cos\phi,$$

and $$CM^2 = b^2 = r^2 + c^2 + 2cr\cos\phi,$$

whence $$CQ = r + \frac{a}{cr}(b^2 - c^2 - r^2),$$

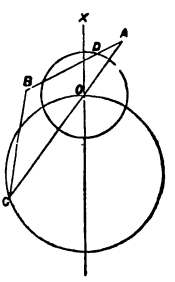
or if Q describe a circle through C, so that $CQ = 2m\cos\theta$,

$$2cmr\cos\theta = r^2(c-a) + a(b^2 - c^2),$$

so that the locus of P is a circle.

II. *Solution by* J. J. WALKER, M.A.

It is plain that the diagonal joining the two angles must pass through the centre of the first circle. The question may therefore be considered as that of finding the locus of one end (A) of the base (AC) of an isosceles triangle ABC, the equal sides (AB, BC) of which are of constant length (a), subject to the conditions that the base shall pass through the centre (O) of one circle, one end (C) move on the circumference passing through the centre of the first, while a fixed point (D) on the opposite side moves on the first circumference.



Let $AD = a'$, $OA = \rho$, $OC = \rho'$, while r, r' represent the radii of the two circles, and θ the angle which ρ makes with the line joining their centres. The following equations are at once verified :—

$$(OD^2 =)\ r^2 = a'^2 + \rho^2 - 2a'\rho\cos A, \quad \rho + \rho' = 2a\cos A, \quad \rho' = 2r'\cos\theta.$$

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Eliminating cos A, and ρ' among these equations, there results for the polar equation of the locus of A, the equation

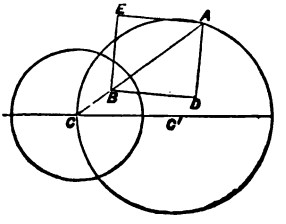
$$(a-a')\rho^2-2a'r'\rho\cos\theta+a(a'^2-r^2)=0,$$

which is that of a circle whose centre (O′) is collinear with those of the given circles, while $OO'=\frac{a'r'}{a-a'}$, and the radius (R) is given by

$$R^2=\frac{a'^2r'^2+a(a-a')(r^2-a'^2)}{(a-a')^2}.$$

It is easy to see when the locus will degenerate into a fixed point; when into a right line perpendicular to that joining the centres of the given circles; and when become imaginary.

III. *Solution by* J. C. MALET, M.A.; A. RENSHAW; *and others.*



Let C, C′ be the centres of the two circles, and BDAE the jointed rhomb. Let r and r' be the radii of the circles; b the constant length intercepted between B and the first circle; and put BD = a, $\angle EBA=\phi$, $\angle BCC'=\theta$, CB = ρ. Now AB must evidently pass through C, and AC = $2r'\cos\theta$; hence we have

$$2r'^2\cos\theta-\rho=2a\cos\phi,$$
$$r^2=\rho^2+b^2+2\rho b\cos\phi,$$

and from these equations we get

$$r^2-b^2=\rho^2+2b\rho\,.\,\frac{2r'\cos\theta-\rho}{2a},\text{ or }(r^2-b^2)\left(\frac{a}{a-b}\right)=\rho^2+2\left(\frac{b}{a-b}\right)\rho r'\cos\theta,$$

which is the equation of a circle whose centre lies on the line CC′ at a distance from C equal to $\frac{br'}{b-a}$, and the square of the radius is equal to $\frac{b^2r'^2}{(b-a)^2}-\frac{a(r^2-b^2)}{b-a}$.

If $a=b$, the locus is a straight line. (See Question 4231.)

If the second circle does not pass through the centre of the first, in place of the relation AC = $2r'\cos\theta$, we have

$$r'^2-(CC')^2=(AC)^2-2(CC')(AC)\cos\theta,$$

and the equation of the locus in Cartesian coordinates is easily found to be

$$(x^2+y^2)^2(a-b)^2+2bdx(x^2+y^2)(a-b)$$
$$-(x^2+y^2)\left\{2a(a-b)(r^2-b^2)+b^2(r'^2-d^2)\right\}-2abdx(r^2-b^2)+a^2(r^2-b^2)^2=0,$$

where d is the distance between the centres of the circles.

When $a=b$, or the first circle passes through D and E, the locus reduces to the circle $(x^2+y^2)b^2(r'^2-d^2)+2abdx(r^2-b^2)=a^2(r^2-b^2)^2$.

[Other Solutions are given on pp. 58—60 of this Volume of the *Reprint*.]

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